Minimality of toric arrangements

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Abstract

We prove that the complement of a toric arrangement has the homotopy type of a minimal CW complex. As a corollary we obtain that the integer cohomology of these spaces is torsion free.

We use Discrete Morse Theory, providing a sequence of cellular collapses that leads to a minimal complex.

Introduction

A *toric arrangement* is a finite family

$$\mathscr{A} = \{K_1, \dots, K_n\}$$

of special subtori of the complex torus $(\mathbb{C}^*)^d$ (more precisely the K_i are level sets of characters, see §2.1). A toric arrangement is called *complexified* if it restricts to an arrangement of level sets of characters of the compact torus $(S^1)^d$. Given a complexified toric arrangement $\mathscr{A}$ we consider the space

$$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \bigcup \mathscr{A}$$

and prove that

(a) the space $M(\mathscr{A})$ is *minimal* in the sense of [13], i.e., it has the homotopy type of a CW complex with exactly $\beta_k = \operatorname{rk} H^k(M(\mathscr{A}); \mathbb{Z})$ cells in dimension k, for every $k \in \mathbb{N}$, hence

(b) the space $M(\mathscr{A})$ is *torsion-free*, that is, the homology and cohomology modules $H_k(X; \mathbb{Z})$, $H^k(X; \mathbb{Z})$ are torsion free for every $k \in \mathbb{N}$.
As a consequence, the cohomology algebra $H^*(M(\mathscr{A}); \mathbb{Z})$ can be derived from a presentation of $H^*(M(\mathscr{A}); \mathbb{C})$.

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The study of toric arrangements experienced a fresh impulse from recent work of De Concini, Procesi and Vergne [10, 9], in which toric arrangements emerge as a link between partition functions and box splines.

In their book [9], De Concini and Procesi emphasize some similarities between toric arrangements and the well-established theory of arrangements of affine hyperplanes.

The present work provides substantial new evidence in this sense. Moreover, it highlights how conveniently some key combinatorial invariants of the topology of toric arrangements can be expressed using tools from the theory of hyperplane arrangements. In particular, a deep interplay between combinatorics and topology seems to permeate both fields of research.

Background

Combinatorics. The combinatorial framework for the theory of arrangements of hyperplanes is widely considered to be given by matroid theory, a well-established branch of combinatorics that has proved very useful in this context ever since the seminal work of Zaslavsky [28].

The combinatorial study of toric arrangements has quite recent roots, and is still in search of a full-fledged pertaining theory. From an enumerative point of view, the arithmetic Tutte polynomial introduced by Moci in [20] summarizes previous results by Ehrenborg, Readdy and Slone [14] and of De Concini and Procesi [9]. This initiated the quest for a variation on the concept of matroid that would suit the 'toric' setting and lead Moci and D'Adderio [5] to suggest a theory of arithmetic matroids as a "combinatorialization" of the essential algebraic data of toric arrangements. Arithmetic matroids in fact encode - but, as yet, do not appear to characterize - some of the crucial combinatorial data of toric arrangements, for example the poset of layers (Definition 33). In this context, our work can be seen as exploration of the properties that would be required from a (still lacking) notion of a 'toric oriented matroid'.

Topology An important result in the theory of arrangements of hyperplanes was established by Brieskorn [3], who proved that the integer cohomology of the complement of an arrangement of complex hyperplanes is torsion-free. This allowed Orlik and Solomon to compute the integer cohomology algebra via the deRham complex [22]. It wasn't until 2004 that minimality of complements of complex hyperplane arrangements was proven by Dimca and Papadima in [13] and by Randell in [24], with Morse theoretic arguments. The explicit construction of such a minimal complex was studied by Yoshinaga [27], Salvetti and Settepanella [26] and the second author [12].

The present paper completes a similar circle of ideas for toric arrangements - but in a different chronological order.

To our knowledge, the first result about the topology of toric arrangements was obtained by Looijenga [18] who deduced the Betti numbers of $M(\mathscr{A})$ from a spectral sequence computation. De Concini and Procesi in [8] explicitely expressed the generators of the cohomology modules over $\mathbb{C}$ in terms of local no broken circuit sets and, for the special case of totally unimodular arrangements, were able to compute the cohomological algebra structure. A presentation of the fundamental group $\pi_1(M(\mathscr{A}))$ of complexified toric arrangements was computed by the authors in [6], based on a combinatorially defined polyhedral complex carrying the homotopy type of the complement $M(\mathscr{A})$ called *toric Salvetti complex*. This polyhedral complex is given as the nerve of an acyclic category[1] and was introduced by the authors in [6], generalizing to arbitrary complexified toric arrangements the complex defined by Moci and Settepanella in [21]. Recently, Davis and Settepanella [7] proved vanishing results for cohomology of toric arrangements with coefficients in some particular local systems.

Outline

Here we prove minimality by exhibiting, for a given complexified toric arrangement $\mathscr{A}$, a minimal CW-complex that is homotopy equivalent to $M(\mathscr{A})$. This complex is obtained from the toric Salvetti complex after a sequence of cellular collapses indexed by a discrete Morse function. The construction of the discrete Morse function relies on a stratification of the toric Salvetti complex where strata are counted by 'local no-broken-circuit sets' (Definition 39), which are known to control the Poincaré polynomial of $M(\mathscr{A})$ by [8].

The (topological) boundary maps of the minimal complex can be recovered in principle from the Discrete Morse data. The explicit computation of such boundary maps is in general difficult even in the case of hyperplane arrangements, where a general formula is known only for dimension 2 by the work of Gaiffi and Salvetti [15] We leave the explicit computation of the boundary maps for our toric complex as a future direction of research.

As an application of our methods, in the last section we describe a construction of the minimal complex for complexified affine arrangements of hyperplanes that uses only the intrinsic combinatorics of the arrangement (i.e. its oriented matroid), as an alternative to the method of [26].

We close our introduction with a detailed outline of the paper.

- We begin with Section 1, where we review some known facts about the combinatorics and the topology of hyperplane arrangements and we prove some

[1] For our use of the term 'acyclic category' see Remark 21

preparatory results about linear extensions of posets of regions of real arrangements.

- In Section 2 we give a short introduction to toric arrangements where we collect some results from the existing literature on which our work is built.
- Section 3 breaks the flow of material directly related to toric arrangements in order to develop Discrete Morse Theory for acyclic categories, generalizing the existing theory for posets.
- We approach the core of our work with Section 4, where we introduce a stratification and a related decomposition of the toric Salvetti complex (Definition 69).
- In order to understand the structure of the pieces of the decomposition of the toric Salvetti complex we need to patch together 'local' combinatorial data, which come from the theory of arrangements of hyperplanes. We do this in Section 5 using diagrams of acyclic categories. The main result here is Theorem 71, proving that every stratum is isomorphic (as a polyhedral complex) to the subdivision of the real torus induced by a complexified toric arrangement.
- Our work culminates with Section 6. The keystone is Proposition 90, where we prove the existence of perfect acyclic matchings for the face categories of subdivisions of the compact torus given by toric arrangements. With this, we can apply the Patchwork Lemma of Discrete Morse Theory (in its version for acyclic categories) to our decomposition of the toric Salvetti complex to get an acyclic matching of the whole complex. This matching can be shown to be perfect and thus prescribes a series of cellular collapses leading to a minimal model for the complement of the toric arrangement.
- In Section 7 then we show that our methods can be used to construct a minimal complex for the complement of (finite) complexified arrangements of hyperplanes.

1 Arrangements of hyperplanes

The theory of hyperplane arrangements is an important ingredient in our treatment of toric arrangements. In order to set the stage for the subsequent considerations, we therefore introduce the language and recall some relevant results about hyperplane arrangements. A standard reference for a comprehensive introduction to the subject is [23].

1.1 Generalities

Through this section let V be a finite dimensional vector space over a field $\mathbb{K}$.

An *affine hyperplane* H in V is the level set of a linear functional on V. That is, there is $\alpha \in V^*$ and $a \in \mathbb{K}$ such that $H = \{v \in V \mid \alpha(v) = a\}$. A set of hyperplanes is called *dependent* or *independent* according to whether the corresponding set of elements of V^* is dependent or not.

Definition 1. An *arrangement of hyperplanes* in V is a collection $\mathscr{A}$ of affine hyperplanes in V.

An hyperplane arrangement $\mathscr{A}$ is called *central* if every hyperplane $H \in \mathscr{A}$ is a linear subspace of V; *finite* if $\mathscr{A}$ is finite; *locally finite* if for every $p \in V$ the set $\{H \in \mathscr{A} \mid p \in H\}$ is finite; *real* (or *complex*, or *rational*) if V is a real (or complex, or rational) vector space.

When we will need to define a total order on the elements of a finite arrangement $\mathscr{A}$, we will do this by simply indexing the elements of $\mathscr{A}$, as $\mathscr{A} = \{H_1, \ldots, H_n\}$.

Much of the theory of hyperplane arrangements is devoted to the study of the *complement* of an arrangement $\mathscr{A}$. That is, the space

$$M(\mathscr{A}) := V \setminus \bigcup \mathscr{A}.$$

Definition 2. Let $\mathscr{A}$ be an hyperplane arrangement, the *intersection poset* of $\mathscr{A}$ is the set

$$\mathcal{L}(\mathscr{A}) := \left\{ \bigcap \mathscr{K} \mid \mathscr{K} \subseteq \mathscr{A} \right\} \setminus \{\emptyset\}$$

of all nonempty intersections of elements of $\mathscr{A}$, ordered by *reverse* inclusion - i.e., for $X, Y \in \mathcal{L}(\mathscr{A})$, $X \geq Y$ if $X \subseteq Y$.

Notice that the whole space V is an element of $\mathcal{L}(\mathscr{A})$ (corresponding to the empty intersection), whereas the empty set is not. The intersection poset is a meet-semilattice and for central hyperplane arrangements is a lattice. Then, we speak of *intersection lattice* of $\mathscr{A}$.

Deletion and restriction

Consider an hyperplane arrangement $\mathscr{A}$ in the vector space V and an intersection $X \in \mathcal{L}(\mathscr{A})$. We associate to X two new arrangements:

$$\mathscr{A}_X = \{H \in \mathscr{A} : X \subseteq H\}, \quad \mathscr{A}^X = \{H \cap X : H \in \mathscr{A} \setminus \mathscr{A}_X\}.$$

Notice that $\mathscr{A}_X$ is an arrangement in V, while $\mathscr{A}^X$ is an arrangement on X.

Remark 1. If a total ordering $\mathscr{A} = \{H_1, \ldots, H_n\}$ is defined, then it is clearly inherited by $\mathscr{A}_X$ for every $X \in \mathcal{L}(\mathscr{A})$. On the elements of $\mathscr{A}^X$ a total ordering is induced as follows. For $L \in \mathscr{A}^X$ define

$$_X L := \min\{H \in \mathscr{A} \mid L \subseteq H\}. \tag{1}$$

Then, order $\mathscr{A}^X := \{L_1, \ldots, L_m\}$ so that, for all $1 \leq i < j \leq m$, $_X L_i <\ _X L_J$ in $\mathscr{A}$.

No Broken Circuit sets

In this section let $\mathscr{A}$ be a central hyperplane arrangement.

Definition 3. A *circuit* is a minimal dependent subset $C \subseteq \mathscr{A}$. A *broken circuit* is a subset of the form

$$C \backslash \{\min C\} \subseteq \mathscr{A}$$

obtained from a circuit removing its least element. A *no broken circuit set* (or, for short, an *nbc* set) is a subset $N \subseteq \mathscr{A}$ which does not contain any broken circuit.

Remark 2. An equivalent definition of nbc set is the following. A subset $N = \{H_{i_1}, \ldots, H_{i_k}\} \subseteq \mathscr{A}$ with $i_1 \leq \cdots \leq i_k$ is a *no broken circuit set* if it is independent and there is no $j < i_i$ such that $N \cup \{H_j\}$ is dependent.

Definition 4. We will write $\mathrm{nbc}(\mathscr{A})$ for the set of no broken circuit sets of $\mathscr{A}$ and $\mathrm{nbc}_k(\mathscr{A}) = \{N \in \mathrm{nbc}(\mathscr{A}) \mid |N| = k\}$ for the set of all no broken circuit sets of cardinality k.

1.2 Real arrangements

In this section we consider the case where $\mathscr{A}$ is an arrangement of hyperplanes in $\mathbb{R}^d$ in order to set up some notation and use the real structure to gain some deeper understanding in the combinatorics of no broken circuit sets.

It is not too difficult to verify that the complement $M(\mathscr{A})$ consists of several contractible connected components. These are called *chambers* of $\mathscr{A}$. We write $\mathcal{T}(\mathscr{A})$ for the set of all chambers of $\mathscr{A}$.

Definition 5. Let $\mathscr{A}$ a real arrangement, the set of *faces* of $\mathscr{A}$ is

$$\mathcal{F}(\mathscr{A}) := \{\overline{C} \cap X \mid C \in \mathcal{T}(\mathscr{A}),\ X \in \mathcal{L}(\mathscr{A})\}.$$

When partially ordered by inclusion, $\mathcal{F}(\mathscr{A})$ is called the *face poset* of $\mathscr{A}$.

Remark 3. A face $F \in \mathcal{F}(\mathscr{A})$ is an open subset of $\bigcap\{H \in \mathscr{A} \mid F \subseteq H\}$. By $\overline{F}$ we mean the topological closure of F in $\mathbb{R}^d$.

One of the main enumerative questions about arrangements of hyperplanes in real space asks for the number of chambers of a given hyperplane arrangement. The answer is very elegant and somehow surprising.

Theorem 6 (Zaslavsky [28])**.** Given a real hyperplane arrangement $\mathscr{A}$, the number its chambers is

$$|\mathcal{T}(\mathscr{A})| = |\operatorname{nbc}(\mathscr{A})|.$$

1.2.1 Taking sides

If $\mathscr{A}$ is an arrangement in a real space V, then every hyperplane H is the locus where a linear form $\alpha_H \in V^*$ takes the value a_H. This way we can associate to each $H \in \mathscr{A}$, its *positive* and *negative halfspace*:

$$H^+ = \{x \in V \mid \alpha_H(x) > a_H\}, \qquad H^- = \{x \in V \mid \alpha_H(x) < a_H\}.$$

Remark 4. A choice of a distinguished chamber $B \in \mathcal{T}(\mathscr{A})$ (the 'base chamber') determines a positive and negative side of every hyperplane. Namely, H^+ is the connected component of $V \backslash H$ which contains B, so that $B = \bigcap_{H \in \mathscr{A}} H^+$. Of course, not every assignment of sides gives rise to such an 'all-positive' chamber.

Definition 7. Consider a complexified locally finite arrangement $\mathscr{A}$ with any choice of 'sides' H^+ and H^- for every $H \in \mathscr{A}$.The *sign vector* of a face $F \in \mathcal{F}(\mathscr{A})$ is the function $\gamma_F : \mathscr{A} \to \{-, 0+\}$ defined as:

$$\gamma_F(H) := \begin{cases} + & \text{if } F \subseteq H^+, \\ 0 & \text{if } F \subseteq H, \\ - & \text{if } F \subseteq H^-. \end{cases}$$

When we will need to specify the arrangement $\mathscr{A}$ to which the sign vector refers, we will write $\gamma[\mathscr{A}]_F(H)$ for $\gamma_F(H)$.

Notice that chambers are precisely those faces whose sign vector maps $\mathscr{A}$ to $\{-, +\}$.

Definition 8. Let C_1 and $C_2 \in \mathcal{T}(\mathscr{A})$ be chambers of a real arrangement, and let $B \in \mathcal{T}(\mathscr{A})$ be a distinguished chamber. We will write

$$S(C_1, C_2) := \{H \in \mathscr{A} \mid \gamma_{C_1}(H) \neq \gamma_{C_2}(H)\}$$

for the set of hyperplanes of $\mathscr{A}$ which separate C_1 and C_2.
For all $C_1, C_2 \in \mathcal{T}(\mathscr{A})$ write

$$C_1 \leq C_2 \iff S(C_1, B) \subseteq S(C_2, B).$$

This turns $\mathcal{T}(\mathscr{A})$ into a poset $\mathcal{T}(\mathscr{A})_B$, the *poset of regions* of the arrangement $\mathscr{A}$ with base chamber B.

Remark 5. Let $\mathscr{A}_0$ be a real arrangement and $B \in \mathcal{T}(\mathscr{A}_0)$. Given a subarrangement $\mathscr{A}_1 \subseteq \mathscr{A}_0$, for every chamber $C \in \mathcal{T}(\mathscr{A}_0)$ there is a unique chamber $\widehat{C} \in \mathcal{T}(\mathscr{A}_1)$ with $C \subseteq \widehat{C}$. The correspondence $C \mapsto \widehat{C}$ defines a surjective map

$$\sigma_{\mathscr{A}_1} : \mathcal{T}(\mathscr{A}_0)_B \to \mathcal{T}(\mathscr{A}_1)_{\widehat{B}}$$

such that $C \leq C'$ implies $\sigma_{\mathscr{A}_1}(C) \leq \sigma_{\mathscr{A}_1}(C')$ for all $C, C' \in \mathcal{T}(\mathscr{A}_0)$.

Definition 9. Let $\mathscr{A}_0$ be a real arrangement and let $\succ_0$ denote any total ordering of $\mathcal{T}(\mathscr{A}_0)$. Consider a subarrangement $\mathscr{A}_1 \subseteq \mathscr{A}_0$. The section

$$\mu[\mathscr{A}_1, \mathscr{A}_0] : \mathcal{T}(\mathscr{A}_1) \to \mathcal{T}(\mathscr{A}_0), \quad C \mapsto \min_{\succ_0}\{K \in \mathcal{T}(\mathscr{A}_0) \mid K \subseteq C\}$$

of $\sigma_{\mathscr{A}_1}$ defines a total ordering $\succ_{0,1}$ on $\mathcal{T}(\mathscr{A}_1)$ by

$$C \succ_{0,1} D \iff \mu[\mathscr{A}_1, \mathscr{A}_0](C) \succ_0 \mu[\mathscr{A}_1, \mathscr{A}_0](D)$$

that we call *induced by* $\succ_0$.

Lemma 10. Consider real arrangements $\mathscr{A}_2 \subseteq \mathscr{A}_1 \subseteq \mathscr{A}_0$, a given total ordering $\succ_0$ of $\mathcal{T}(\mathscr{A}_0)$ and the induced total ordering $\succ_{0,1}$ of $\mathcal{T}(\mathscr{A}_1)$. Then

$$\mu[\mathscr{A}_1, \mathscr{A}_0] \circ \mu[\mathscr{A}_2, \mathscr{A}_1] = \mu[\mathscr{A}_2, \mathscr{A}_0].$$

Proof. Take any $C \in \mathcal{T}(\mathscr{A}_2)$ and define

$$\begin{aligned} &C_0 := \mu[\mathscr{A}_2, \mathscr{A}_0](C); \quad C_1 := \sigma_{\mathscr{A}_1}(C_0), \text{ so } \mu[\mathscr{A}_1, \mathscr{A}_0](C_1) = C_0; \\ &C_2 := \mu[\mathscr{A}_2, \mathscr{A}_1](C); \quad C_3 := \mu[\mathscr{A}_1, \mathscr{A}_0](C_2). \end{aligned}$$

we have to show that $C_0 = C_3$.
First, notice that $C_0 \preceq_0 C_3$ because $C_3 \subseteq C_2 \subseteq C$. For the reverse inequality notice that we have $C_1, C_2 \subseteq C$, which implies $C_2 \preceq_{0,1} C_1$ and so, by definition of the induced ordering, $C_3 = \mu[\mathscr{A}_1, \mathscr{A}_0](C_2) \preceq_0 \mu[\mathscr{A}_1, \mathscr{A}_0](C_1) = C_0$. □

Proposition 11. Let a base chamber B of $\mathscr{A}_0$ be chosen. If $\succ_0$ is a linear extension of $\mathcal{T}(\mathscr{A}_0)_B$, then $\succ_{0,1}$ is a linear extension of $\mathcal{T}(\mathscr{A}_1)_{\widehat{B}}$.

Proof. We have to prove that for all $C, D \in \mathcal{T}(\mathscr{A}_1)$, $C \leq D$ in $\mathcal{T}(\mathscr{A}_1)_{\widehat{B}}$ implies $C \preceq_{0,1} D$, i.e., $\mu[\mathscr{A}_0, \mathscr{A}_1](C) \preceq_0 \mu[\mathscr{A}_0, \mathscr{A}_1](D)$.

We argue by induction on $k := |\mathscr{A}_0 \setminus \mathscr{A}_1|$, the claim being evident when $k = 0$. Suppose then that $k > 0$, choose $H \in \mathscr{A}_0 \setminus \mathscr{A}_1$ and set $\mathscr{A}_0' := \mathscr{A}_0 \setminus \{H\}$. By induction hypothesis we have

$$\mu[\mathscr{A}_0', \mathscr{A}_1](C) \preceq_0' \mu[\mathscr{A}_0', \mathscr{A}_1](D).$$

which by definition means

$$\mu[\mathscr{A}_0, \mathscr{A}_0'](\mu[\mathscr{A}_0', \mathscr{A}_1](C)) \preceq_0 \mu[\mathscr{A}_0, \mathscr{A}_0'](\mu[\mathscr{A}_0', \mathscr{A}_1](D))$$

and thus, via Lemma 10, $\mu[\mathscr{A}_0, \mathscr{A}_1](C) \preceq_0 \mu[\mathscr{A}_0, \mathscr{A}_1](D)$. □

1.3 Complex(ified) arrangements

We turn to the case of complex hyperplane arrangements, where the space $M(\mathscr{A})$ has more subtle topology. For the sake of concision here we deliberately disregard the chronological order in which the relevant theorems were proved, and start with the minimality result.

Definition 12. Let X be a topological space. For $j \geq 0$, the *j-th Betti number* is $\beta_j(X) := \operatorname{rk} H^j(M(\mathscr{A}), \mathbb{Z})$. The space X is called *minimal* if it is homotopy equivalent to a CW-complex with $\beta_j(X)$ cells of dimension j, for all $j \geq 0$. Such a CW-complex is also called minimal.

Theorem 13 (Randell [24], Dimca and Papadima [13])**.** The space $M(\mathscr{A})$ is minimal.

Corollary 14. The cohomology groups $H^k(M(\mathscr{A}), \mathbb{Z})$ are torsion-free.

Proof. Theorem 13 asserts the existence of a minimal complex for $M(\mathscr{A})$. The (algebraic) boundary maps of the chain complex constructed from this minimal complex are all zero, thus torsion cannot arise in homology. □

Remark 6. Corollary 14 was in fact one of the very first breakthrough in the theory of arrangements of hyperplanes, and can be aced back to the seminal work of Brieskorn [3], where also the following other important fact about the cohomology of affine arrangements of hyperplanes was proved.

Theorem 15 (Brieskorn [3])**.** Let $\mathscr{A}$ be a finite affine hyperplane arrangement. Then, for every $p \in \mathbb{N}$

$$H^p(M(\mathscr{A}); \mathbb{Z}) \cong \bigoplus_{X \in \mathcal{L}(\mathscr{A})_p} H^p(M(\mathscr{A}_X); \mathbb{Z}),$$

where $\mathcal{L}(\mathscr{A})_p = \{X \in \mathcal{L}(\mathscr{A}) \mid \operatorname{codim}(X) = p\}$.

Intimely related with this torsion-freeness is the fact that it is enough to compute de Rham cohomology in order to know the cohomology with integer coefficients. Here, too, no broken circuit sets enter the picture as most handy combinatorial invariants.

Theorem 16. Let $\mathscr{A}$ be a complex central hyperplane arrangement, then the Poincaré polynomial of $M(\mathscr{A})$ satisfies

$$P_{\mathscr{A}}(t) := \sum_{j=0}^{\infty} \operatorname{rk} H^j(M(\mathscr{A}); \mathbb{Z})\, t^j = \sum_{j=0} |\operatorname{nbc}_j(\mathscr{A})|\, t^j.$$

Remark 7. In particular, the numbers $|\operatorname{nbc}_k(\mathscr{A})|$ do not depend on the chosen ordering of $\mathscr{A}$.

Remark 8 ([16]). Combining Theorem 15 with Theorem 16 we get the following formula for the Poincaré polynomial of the complement of an arbitrary finite affine complex arrangement:

$$P_{\mathscr{A}}(t) := \sum_{X \in \mathcal{L}(\mathscr{A})} |\operatorname{nbc}_{\operatorname{codim} X}(\mathscr{A}_X)|\, t^{\operatorname{codim} X}.$$

We now turn to a special class of arrangements in complex space.

Definition 17. An arrangement $\mathscr{A}$ in $\mathbb{C}^d$ is called *complexified* if every hyperplane $H \in \mathscr{A}$ is the complexification of a real hyperplane, i.e. if there is $\alpha_H \in (\mathbb{R}^d)^*$ and $a_H \in \mathbb{R}$ with

$$H = \{x \in \mathbb{C}^d \mid \alpha_H(\Re(x)) + i\alpha_H(\Im(x)) = a_H\}.$$

Let $\mathscr{A}$ be a complexified arrangement and consider its real part

$$\mathscr{A}_{\mathbb{R}} = \{H \cap \mathbb{R}^d \mid H \in \mathscr{A}\},$$

an arrangement of hyperplanes in $\mathbb{R}^d$. Notice that $\mathcal{L}(\mathscr{A}) \cong \mathcal{L}(\mathscr{A}_{\mathbb{R}})$ and therefore $\operatorname{nbc}(\mathscr{A}) = \operatorname{nbc}(\mathscr{A}_{\mathbb{R}})$.

If $\mathscr{A}$ is a complexified arrangement, one can use the combinatorial structure of $\mathscr{A}_{\mathbb{R}}$ to study the topology of $M(\mathscr{A})$. Therefore we will write $\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathscr{A}_{\mathbb{R}})$, $\mathcal{T}(\mathscr{A}) = \mathcal{T}(\mathscr{A}_{\mathbb{R}})$.

The homotopy type of complexified arrangements

Using combinatorial data about $\mathscr{A}_{\mathbb{R}}$, Salvetti defined in [25] a cell complex which embeds in the complement $M(\mathscr{A})$ as a deformation retract. We explain Salvetti's construction.

Definition 18. Let $F \in \mathcal{F}(\mathscr{A})$ face and $C \in \mathcal{T}(\mathscr{A})$ chamber, define the chamber $C_F \in \mathcal{T}(\mathscr{A})$ as the unique chamber such that

$$\gamma_{C_F}(H) = \begin{cases} \gamma_F(H) & \text{if } \gamma_F(H) \neq 0 \\ \gamma_C(H) & \text{if } \gamma_F(H) = 0 \end{cases}$$

The reader may think of C_F as the one, among the chambers adjacent to F, that "faces" C.

Definition 19. Consider an affine complexified locally finite arrangement $\mathscr{A}$ and define the *Salvetti poset* as follows:

$$\operatorname{Sal}(\mathscr{A}) = \{[F, C] \mid F \in \mathcal{F}(\mathscr{A}), C \in \mathcal{T}(\mathscr{A})\ F \leq C\},$$

with the relation

$$[F_1, C_1] \leq [F_2, C_2] \iff F_2 \leq F_1 \text{ and } (C_2)_{F_1} = C_1.$$

Definition 20. Let $\mathscr{A}$ be an affine complexified locally finite hyperplane arrangement. Its *Salvetti complex* is $\mathcal{S}(\mathscr{A}) = \Delta(\mathrm{Sal}(\mathscr{A}))$.

Theorem 21 (Salvetti [25])**.** The complex $\mathcal{S}(\mathscr{A})$ is homotopically equivalent to the complement $M(\mathscr{A})$. More precisely $\mathcal{S}(\mathscr{A})$ embeds in $M(\mathscr{A})$ as a deformation retract.

Remark 9. In fact, the poset $\mathrm{Sal}(\mathscr{A})$ is the face poset of a regular cell complex (of which $\mathcal{S}(\mathscr{A})$ is the barycentric subdivision) whose maximal cells correspond to the pairs

$$\{[P, C] \mid P \in \min \mathcal{F}(\mathscr{A}),\, C \in \mathcal{T}(\mathscr{A})\}.$$

It is this complex that Salvetti describes in [25]. When we need to distinguish between the two complexes we will speak of *cellular* and *simplicial Salvetti complex*.

Minimality

In the case of complexified arrangements, explicit constructions of a minimal CW-complex for $M(\mathscr{A})$ were given in [26] and in [12]. We review the material of [12, §4] that will be useful for our later purposes.

Lemma 22 ([12, Theorem 4.13])**.** Let $\mathscr{A}$ be a central arrangement of real hyperplanes, let $B \in \mathcal{T}(\mathscr{A})$ and let $\preceq$ be any linear extension of the poset $\mathcal{T}(\mathscr{A})_B$. The subset of $\mathcal{L}(\mathscr{A})$ given by all intersections X such that

$$S(C, C') \cap \mathscr{A}_X \neq \emptyset \quad \text{for all } C' \prec C$$

is an order ideal of $\mathcal{L}(\mathscr{A})$. In particular, it has a well defined and unique minimal element we will call X_C.

Remark 10. Note that X_C depends on the choice of B and of the linear extension of $\mathcal{T}(\mathscr{A})_B$.

Corollary 23. For all $C \in \mathcal{T}(\mathscr{A})$ we have

$$C = \min_{\preceq}\{K \in \mathcal{T}(\mathscr{A}) \mid K_{X_C} = C_{X_C}\},$$

where, for $Y \in \mathcal{L}(\mathscr{A})$ and $K \in \mathcal{T}(\mathscr{A})$, we define $K_Y := \sigma_{\mathscr{A}_Y}(K)$.

Now recall the (cellular) Salvetti complex of Definition 20 and Remark 9. In particular, its maximal cells correspond to the pairs $[P, C]$ where P is a point and C is a chamber. When $\mathscr{A}$ is a central arrangement, the maximal cells correspond to the chambers in $\mathcal{T}(\mathscr{A})$. In this case we can stratify the Salvetti complex assigning to each chamber $C \in \mathcal{T}(\mathscr{A})$ the corresponding maximal cell of $\mathcal{S}(\mathscr{A})$, together with its faces. Let us make this precise.

Definition 24. Let $\mathscr{A}$ be a central complexified hyperplane arrangement and write $\min \mathcal{F}(\mathscr{A}) = \{P\}$. Define a stratification of the cellular Salvetti complex $\mathcal{S}(\mathscr{A}) = \bigcup_{C\in\mathcal{T}(\mathscr{A})} \mathcal{S}_C$ through

$$\mathcal{S}_C := \bigcup \{[F,K] \in \mathrm{Sal}(\mathscr{A}) \mid [F,K] \leq [P,C]\} .$$

Given an arbitrary linear extension $(\mathcal{T}(\mathscr{A}), \preceq)$ of $\mathcal{T}(\mathscr{A})_B$, for all $C \in \mathcal{T}(\mathscr{A})$ define

$$\mathcal{N}_C := \mathcal{S}_C \setminus \left(\bigcup_{D\prec C} \mathcal{S}_D \right).$$

In particular the poset $\mathrm{Sal}(\mathscr{A})$ can be partitioned as

$$\mathrm{Sal}(\mathscr{A}) = \bigsqcup_{C\in\mathcal{T}(\mathscr{A})} \mathcal{N}_C(\mathscr{A}).$$

Theorem 25 ([12, Lemma 4.18])**.** There is an isomorphism of posets

$$\mathcal{N}_C \cong \mathcal{F}(\mathscr{A}^{X_C})^{op}$$

where X_C is the intersection defined via Lemma 22 by the same choice of base chamber and of linear extension of $\mathcal{T}(\mathscr{A})_B$ used to define the subposets $\mathcal{N}_C$.

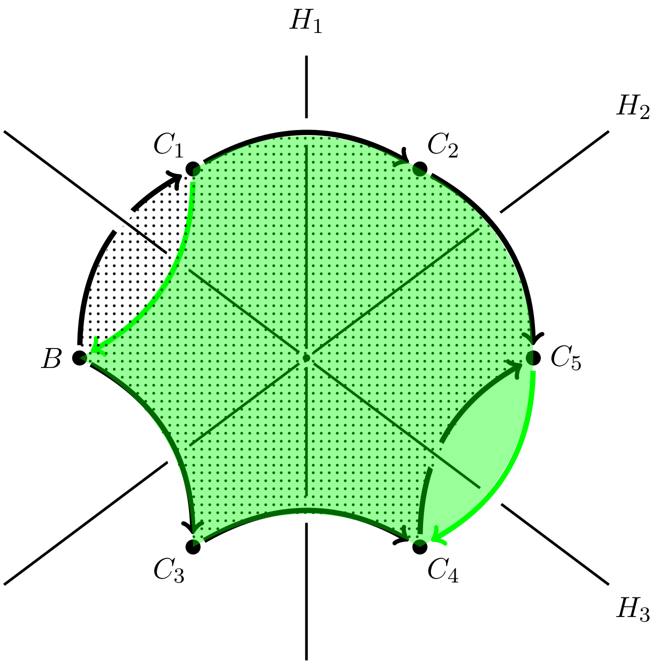
Remark 11. The alternative proof given in [12] of minimality of $M(\mathscr{A})$ for $\mathscr{A}$ a complexified central arrangement follows from Theorem 25 by an application of Discrete Morse Theory (see Section 3). Indeed, from a shelling order of $\mathcal{F}(\mathscr{A}^{X_C})$ one can construct a sequence of cellular collapses of the induced subcomplex of $\mathcal{S}_C$ that leaves only one 'surviving' cell. Via the Patchwork Lemma (Lemma 52 below) these sequences of collapses can be concatenated to give a sequence of collapses on the cell complex $\mathcal{S}(\mathscr{A})$. The resulting complex after the collapses has one cell for every $\mathcal{N}_C$, namely $|\,\mathrm{nbc}(\mathcal{A})| = P_{\mathscr{A}}(1)$ cells, and is thus minimal.

Example 26. Consider the arrangement of Figure 1. We have

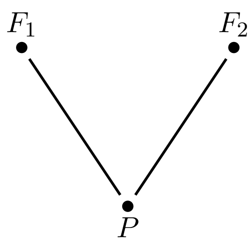
$$\mathcal{L}(\mathscr{A}) = \{\mathbb{R}^2, H_1, H_2, H_3, P\}$$

where $P = H_1 \cap H_2 \cap H_3$. The chambers are ordered according to their indices, B being the base chamber. Then, $X_B = \mathbb{R}^2$, $X_{C_1} = H_3$, $X_{C_2} = H_1$, $X_{C_3} = H_2$, $X_{C_4} = X_{C_5} = P$.

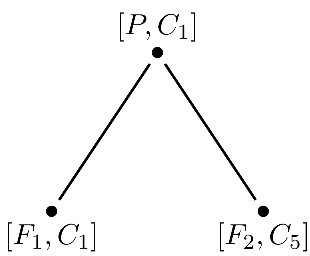
Recall the construction of the *cellular* Salvetti Complex (e.g. from [6, Definition 2.4]). Figure 1.(a) shows in dotted black the stratum $\mathcal{S}_B = \mathcal{N}_B$ and in solid green the stratum $\mathcal{N}_{C_1}$. The stratum $\mathcal{N}_{C_1}$ consists of two 1-dimensional faces and one 2-dimensional face. Its poset structure is showed in Figure 1.(c) and it is isomorphic, as a poset, to the rder dual of $\mathcal{F}(\mathscr{A}^{X_{C_1}})$, depicted in Figure 1.(b).



(a) $\mathcal{S}_B$ and $\mathcal{N}_{C_1}$



(b) $\mathcal{F}(\mathscr{A}^{X_{C_1}})$



(c) $\mathcal{N}_{C_1}$

Figure 1: Example of stratification

2 Toric arrangements

2.1 Introduction

We have presented arrangements of hyperplanes in affine space as families of level sets of linear forms. Now, we want to explain in which sense this idea has been generalized to a toric setting.

Our ambient spaces will be the *complex torus* $(\mathbb{C}^*)^d$ and the *compact* (or *real*) *torus* $(S^1)^d$, where we consider S^1 as the unit circle in $\mathbb{C}$. We consider *characters* of the torus, i.e., maps $\chi : (\mathbb{C}^*)^d \to \mathbb{C}^*$ given by

$$\chi(x_1, \dots, x_d) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} \text{ for an } \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d.$$

Characters form a lattice, which we denote by Λ, under pointwise multiplication. Notice that the assignment $\alpha \mapsto x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ provides an isomorphism $\mathbb{Z}^d \to \Lambda$.

We consider subtori defined as level sets of characters, that is hypersurfaces in $(\mathbb{C}^*)^d$ of the form

$$K = \{x \in (\mathbb{C}^*)^d \mid \chi(x) = a\} \text{ with } \chi \in \Lambda, a \in \mathbb{C}^*. \tag{2}$$

Notice that, if $a \in S^1$, the interesection $K \cap (S^1)^d$ is also a level set of a character (described by the same equation).

Definition 27. A *(complex) toric arrangement* $\mathscr{A}$ in $(\mathbb{C}^*)^d$ is a finite set

$$\mathscr{A} = \{K_1, \dots, K_n\}$$

of hypersufaces of the form (2) in $(\mathbb{C}^*)^d$

Definition 28. Let $\mathscr{A}$ be a toric arrangement in $(\mathbb{C}^*)^d$. Its *complement* is

$$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \bigcup \mathscr{A}.$$

Definition 29. A *real toric arrangement* $\mathscr{A}$ in $(S^1)^d$ is a finite set

$$\mathscr{A}^c = \{K_1^c, \dots, K_n^c\}$$

of hypersufaces in $(S^1)^d$ of the form (2) with $a \in S^1$. If a complex toric arrangement restricts to a real toric arrangement on $(S^1)^d$ we will call $\mathscr{A}$ *complexified.*

We will often use this interplay between the complex and the 'real' hypersurfaces in the same vein that one exploits properties of the real part of complexified arrangements to gain insight into the complexification.

2.2 An abstract approach

We now introduce an equivalent but more abstract approach to toric arrangements. Being able to switch point of view according to the situation will make our considerations below considerably more transparent.

Definition 30. Let $\Lambda \cong \mathbb{Z}^d$ a finite rank lattice. The corresponding *complex torus* is

$$T_\Lambda = \hom_{\mathbb{Z}}(\Lambda, \mathbb{C}^*).$$

The *compact (or real) torus* corresponding to Λ is

$$T_\Lambda^c = \hom_{\mathbb{Z}}(\Lambda, S^1),$$

where, again, $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$.

The choice of a basis $\{v_1, \ldots, v_d\}$ of Λ gives isomorphisms

$$\begin{array}{cc} \Phi : T_\Lambda \to (\mathbb{C}^*)^d & \Phi^c : T_\Lambda^c \to (S^1)^d \\ \varphi \mapsto (\varphi(v_1), \ldots, \varphi(v_d)) & \varphi \mapsto (\varphi(v_1), \ldots, \varphi(v_d)) \end{array} \tag{3}$$

Remark 12. Consider a finite rank lattice Λ and the corresponding torus T_Λ. The *characters* of T_Λ are the functions

$$\chi_\lambda : T_\Lambda \to \mathbb{C}^*, \quad \chi_\lambda(\varphi) = \varphi(\lambda) \text{ with } \lambda \in \Lambda.$$

Characters form a lattice under pointwise multiplication, and this lattice is naturally isomorphic to Λ. Therefore in the following we will identify the character lattice of T_Λ with Λ.

Now, the 'abstract' definition of toric arrangements is the following.

Definition 31. Consider a finite rank lattice Λ, a *toric arrangement* in T_Λ is a finite set of pairs

$$\mathscr{A} = \{(\chi_1, a_1), \ldots (\chi_n, a_n)\} \subset \Lambda \times \mathbb{C}^*.$$

A toric arrangement $\mathscr{A}$ is called *complexified* if $\mathscr{A} \subset \Lambda \times S^1$.

Remark 13. The abstract definition is clearly equivalent to Definition 29 via the isomorphisms in (3) and by

$$K_i := \{x \in T_\Lambda \mid \chi_i(x) = a_i\}. \tag{4}$$

Accordingly, we have $M(\mathscr{A}) := T_\Lambda \setminus \bigcup\{K_1, \ldots, K_n\}$.

Definition 32. Let Λ be a finite rank lattice. A *real* toric arrangement in T_Λ^c is a finite set of pairs

$$\mathscr{A}^c = \{(\chi_1, a_1), \ldots (\chi_n, a_n)\} \subset \Lambda \times S^1.$$

Remark 14. A complexified toric arrangement $\mathscr{A}$ in T_Λ induces a real toric arrangement $\mathscr{A}^c$ in T_Λ^c with

$$K_i^c := \{x \in T_\Lambda^c \mid \chi_i(x) = a_i\}.$$

Furthermore, embedding $T_\Lambda^c \hookrightarrow T_\Lambda$ in the obvious way, we have $K_i^c = K_i \cap T_\Lambda^c$ as in Definition 29 .

We now illustrate what has been proposed [8, 19] as the 'toric analogue' of the intersection poset.

Definition 33. Let $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$ be a toric arrangement on T_Λ. A *layer* of $\mathscr{A}$ is a connected component of a nonempty intersection of some of the subtori K_i (defined in Remark 13). The set of all layers of $\mathscr{A}$ ordered by reverse inclusion is the *poset of layers* of the toric arrangement, denoted by $\mathcal{C}(\mathscr{A})$.

Notice that, as for hyperplane arrangements, the torus T_Λ itself is a layer, while the empty set is not.

Definition 34. Let Λ be a rank d lattice and let $\mathscr{A}$ be a toric arrangement on T_Λ. The *rank* of $\mathscr{A}$ is $\mathrm{rk}\,(\mathscr{A}) := \mathrm{rk}\,\langle \chi \mid (\chi, a) \in \mathscr{A}\rangle$

(a) A character $\chi \in \Lambda$ is called *primitive* if, for all $\psi \in \Lambda$, $\chi = \psi^k$ only if $k \in \{-1, 1\}$.

(b) The toric arrangement $\mathscr{A}$ is called *primitive* if for each $(\chi, a) \in \mathscr{A}$, χ is primitive.

(c) The toric arrangement $\mathscr{A}$ is called *essential* if $\mathrm{rk}\,(\mathscr{A}) = d$.

Remark 15. For every non primitive arrangement there is a primitive arrangement which has the same complement. Furthermore, if $\mathscr{A}$ is a non essential arrangement, then there exist an essential arrangement $\mathscr{A}'$ such that

$$M(\mathscr{A}) \cong (\mathbb{C}^*)^{d-l} \times M(\mathscr{A}') \text{ where } l = \mathrm{rk}\,(\mathscr{A}').$$

Therefore the topology of $M(\mathscr{A})$ can be derived from the topology of $M(\mathscr{A}')$.

In view of Remark 15, our study of the topology of complements of toric arrangements will not loose in generality by stipulating the next assumption.

Assumption 35. From now on we assume every toric arrangement to be *primitive* and *essential.*

2.2.1 Deletion and restriction

Let Λ be a finite rank lattice and $\mathscr{A}$ be a toric arrangement in T_Λ.

Definition 36. For every sublattice $\Gamma \subseteq \Lambda$ we define the arrangement

$$\mathscr{A}_\Gamma = \{(\chi, a) \mid \chi \in \Gamma\},$$

for every layer $X \in \mathcal{C}(\mathscr{A})$ a sublattice

$$\Gamma_X := \{\chi \in \Lambda \mid \chi \text{ is constant on } X\} \subseteq \Lambda.$$

Definition 37. Let X be a layer of $\mathscr{A}$. We define toric arrangements

$$\mathscr{A}_X := \mathscr{A}_{\Gamma_X} \text{ on } T_{\Gamma_X},$$

and

$$\mathscr{A}^X := \{K_i \cap X \mid X \not\subseteq K_i\} \text{ on the torus } X.$$

Remark 16. Notice that for a layer $X \in \mathcal{C}(\mathscr{A})$ and an hypersurface K of $\mathscr{A}$, the interesection $K \cap X$ needs not to be connected.

In general $K \cap X$ consist of several connected components, each of which is a level set of a character in the torus X. In particular $\mathscr{A}^X$ is a toric arrangement in the sense of Definition 31

2.2.2 Covering space

We now recall a construction of [6] which we need in the following. For more details we refer to [6, §3.2]. Consider the covering map:

$$\begin{gathered} p : \mathbb{C}^d \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C}^*) = T_\Lambda \\ \varphi \mapsto \exp \circ \varphi \end{gathered} \tag{5}$$

Notice that identifying $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \cong \mathbb{C}^d$, p becomes the universal covering map

$$(t_1, \ldots, t_d) \mapsto (e^{2\pi i t_1}, \cdots, e^{2\pi i t_d})$$

of the torus T_Λ. Also, this map restricts to a universal covering map

$$\mathbb{R}^d \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, S^1) \cong (S^1)^d.$$

Consider now a toric arrangement $\mathscr{A}$ on T_Λ. Its preimage through p is a locally finite affine hyperplane arrangement on $\mathrm{Hom}_{\mathbb{Z}}(\Lambda; \mathbb{C})$

$$\mathscr{A}^\upharpoonright = \{(\chi, a') \in \Lambda \times \mathbb{C} \mid (\chi, e^{2\pi i a'}) \in \mathscr{A}\}.$$

If we write it in coordinates, $\mathscr{A}^\upharpoonright$ becomes the arrangement on $\mathbb{C}^d$ defined as

$$\mathscr{A}^\upharpoonright = \{H_{\chi,a'} \mid (\chi, e^{2\pi i a'}) \in \mathscr{A}\} \text{ with } H_{\chi,a'} = \{x \in \mathbb{C}^n \mid \sum \alpha_i x_i = a'\},$$

where we expanded $\chi(x) = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$.

Remark 17. If the toric arrangement $\mathscr{A}$ is complexified, so is the hyperplane arrangement $\mathscr{A}^\upharpoonright$.

2.3 Combinatorics

As in the case of hyperplanes, one would like to describe the topology of the complement in terms of the combinatorics of the arrangement.

Lemma 38. Let $\mathscr{A}$ be a toric arrangement, $X \in \mathcal{C}(\mathscr{A})$ a layer. Then the subposet $\mathcal{C}(\mathscr{A})_{\leq X}$ is the intersection poset of a central hyperplane arrangement $\mathscr{A}[X]$. If $\mathscr{A}$ is complexified, then $\mathscr{A}[X]$ is, too.

Proof. This is implicit in much of [8, 19], the proof follows by lifting the layer X to $\mathscr{A}^{\upharpoonright}$. A formally precise definition of $\mathscr{A}[Y]$ can also be found in Section 4.1 below. □

In other words, lower intervals of posets of layers are intersection lattices of (central) hyperplane arrangements. The following definition is then natural.

Definition 39 ([8, 19])**.** Let $\mathscr{A}$ be a toric arrangement of rank d and let us fix a total ordering on $\mathscr{A}$. A *local no broken circuit set* of $\mathscr{A}$ is a pair

$$(X, N) \text{ with } X \in \mathcal{C}(\mathscr{A}), N \in \mathrm{nbc}_k(\mathscr{A}(X)) \text{ where } k = d - \dim X$$

We will write $\mathscr{N}$ for the set of local non broken circuits, and partition it into subsets

$$\mathscr{N}_j = \{(X, N) \in \mathscr{N} \mid \dim X = d - j\}.$$

Remark 18. Let $X \in \mathcal{C}(\mathscr{A})$ and $N \subseteq \mathscr{A}(X)$. If we consider the 'list' $\mathscr{X}$ of all pairs (χ_i, a_i) with $\chi_{i|X} \equiv a_i$, then the elements of N index a 'sublist' $\mathscr{X}_N$. Then, (X, N) is a local no broken circuit set if and only if $\mathscr{X}_N$ is a basis of $\mathscr{X}$ with no *local external activity* in the sense of d'Adderio and Moci [5, Section 5.3]

2.4 Cohomology

The cohomology (with complex coefficients) of the complements of toric arrangements was studied by Looijenga [18] and De Concini and Procesi [8].

Theorem 40 ([8, Theorem 4.2])**.** Consider a toric arrangement $\mathscr{A}$. The Poincaré polynomial of $M(\mathscr{A})$ can be expressed as follows:

$$P_{\mathscr{A}}(t) = \sum_{j=0}^{\infty} \dim H^j(M(\mathscr{A}); \mathbb{C})\, t^j = \sum_{j=0}^{\infty} |\mathscr{N}_j|\, (t+1)^{k-j}\, t^j.$$

This result was reached in [8] by computing de Rham cohomology, in [18] via spectral sequence computations. In the special case of (totally) unimodular arrangements, De Concini and Procesi also determine the algebra structure of $H^*(M(\mathscr{A}), \mathbb{C})$ by formality of $M(\mathscr{A})$ [8, Section 5].

2.5 The homotopy type of complexified toric arrangements

From now on in this paper we will think of $\mathscr{A}$ as being a complexified (primitive, essential) toric arrangement.

The complement of a complexified toric arrangement $\mathscr{A}$ has the homotopy type of a finite cell complex, defined from the stratification of the real torus T_Λ into *chambers* and *faces* induced by the associated 'real' arrangement $\mathscr{A}^c$.

Definition 41. Consider a complexified toric arrangement $\mathscr{A} = \{(\chi_1, a_1), \dots, (\chi_n, a_n)\}$, its *chambers* are the connected components of $M(\mathscr{A}^c)$. We denote the set of chambers of $\mathscr{A}$ by $\mathcal{T}(\mathscr{A})$.

The *faces* of $\mathscr{A}$ are the connected compontent of the intersections:

$$\overline{C} \cap X \text{ with } C \in \mathcal{T}(\mathscr{A})\; X \in \mathcal{C}(\mathscr{A}).$$

The faces of $\mathscr{A}$ are the cells of a polyhedral complex, which we denote by $\mathcal{D}(\mathscr{A})$.

The topology of a (non regular) polyedral complex is encoded in an acyclic category, called the *face category* of the complex (see [6, §2.2.2] for some details on face categories and [17] for details on acyclic categories).

Definition 42. The face category of a complexified toric arrangement is $\mathcal{F}(\mathscr{A}) = \mathcal{F}(\mathcal{D}(\mathscr{A}))$, i.e. the face category of the polyhedral complex $\mathcal{D}(\mathscr{A})$.

The lattice Λ acts on $\mathbb{C}^n$ and on $\mathbb{R}^n$ as the group of automorphisms of the covering map p of (5) above. Consider now the map $q : \mathcal{F}(\mathscr{A}^\upharpoonright) \to \mathcal{F}(\mathscr{A})$ induced by p.

Proposition 43 ([6, Lemma 4.8])**.** Let $\mathscr{A}$ be a complexified toric arrangement. The map $q : \mathcal{F}(\mathscr{A}^\upharpoonright) \to \mathcal{F}(\mathscr{A})$ induces an isomorphism of acylic categories

$$\mathcal{F}(\mathscr{A}) \cong \mathcal{F}(\mathscr{A}^\upharpoonright)/\Lambda.$$

2.5.1 The Salvetti category

Recall that the Salvetti complex for affine hyperplane arrangements makes use of the operation of Definition 18. We need a suitable analogon for toric arrangements.

Proposition 44 ([6, Proposition 3.12])**.** Let Λ be a finite rank lattice, Γ a sublattice of Λ. Let $\mathscr{A}$ a complexfied toric arrangement on T_Λ and recall the arrangement $\mathscr{A}_\Gamma$ from Definition 36. The projection $\pi_\Gamma : T_\Lambda \to T_\Gamma$ induces a morphism of acyclic categories

$$\pi_\Gamma : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_\Gamma).$$

Consider now a face $F \in \mathcal{F}(\mathscr{A})$. We associate to it the sublattice

$$\Gamma_F = \{\chi \in \Lambda \mid \chi \text{ is constant on } F\} \subseteq \Lambda$$

Definition 45. Consider a toric arrangement $\mathscr{A}$ on T_Λ and a face $F \in \mathcal{F}(\mathscr{A})$. The *restriction* of $\mathscr{A}$ to F is the arrangement $\mathscr{A}_F = \mathscr{A}_{\Gamma_F}$ on T_{Γ_F}.

We will write $\pi_F = \pi_{\Gamma_F} : \mathcal{F}(\mathscr{A}) \to \mathcal{F}(\mathscr{A}_F)$.

Definition 46 ([6, Definition 4.1])**.** Let $\mathscr{A}$ be a toric a arrangement on a complex torus T_Λ. The *Salvetti category* of $\mathscr{A}$ is the category $\mathrm{Sal}\,\mathscr{A}$ defined as follows.

(a) The objects are the morphisms in $\mathcal{F}(\mathscr{A})$ between faces and chambers:

$$\mathrm{Obj}(\mathrm{Sal}\,\mathscr{A}) = \{m : F \to C \mid m \in \mathrm{Mor}(\mathcal{F}(\mathscr{A})),\, C \in \mathcal{T}(\mathscr{A})\}.$$

(b) The morphisms are the triples $(n, m_1, m_2) : m_1 \to m_2$, where $m_1 : F_1 \to C_1, m_2 : F_2 \to C_2 \in \mathrm{Obj}(\mathrm{Sal}\,\mathscr{A})$, $n : F_2 \to F_1 \in \mathrm{Mor}(\mathcal{F}(\mathscr{A}))$ and m_1, m_2 satisfy the condition:
$$\pi_{F_1}(m_1) = \pi_{F_1}(m_2).$$

(c) Composition of morphisms is defined as:

$$(n', m_2, m_3) \circ (n, m_1, m_2) = (n \circ n', m_1, m_3),$$

whenever n and n' are composable.

Remark 19. The Salvetti category is an acyclic category in the sense of Definition 49.

Definition 47. Let $\mathscr{A}$ be a complexified toric arrangement; its *Salvetti complex* is the nerve $\mathcal{S}(\mathscr{A}) = \Delta(\mathrm{Sal}\,\mathscr{A})$.

Theorem 48 ([6, Theorem 4.3])**.** The Salvetti complex $\mathcal{S}(\mathscr{A})$ embeds in the complement $M(\mathscr{A})$ as a deformation retract.

Remark 20. As for the case of affine arrangements, the Salvetti category is the face category of a polyhedral complex, of which the toric Salvetti complex is a subdivision. If we need to distinguish between the two, we will call the first *cellular Salvetti complex* and the second *simplicial Salvetti complex.*

3 Discrete Morse theory

Our proof of minimality will consist in describing a sequence of cellular collapses on the toric Salvetti complex, which is not necessarily a regular cell complex. We need thus to estend discrete Morse theory from posets to acyclic categories.

The setup used in the textbook of Kozlov [17] happens to lend itself very nicely to such a generalization - in fact, once the right definitions are made, even the proofs given in [17] just need some minor additional observation.

Definition 49. An *acyclic category* is a small category where the only endomorphisms are the identities, and these are the only invertible morphisms.

An *indecomposable morphism* in an acyclic category is a morphism that cannot be written as the composition of two nontrivial morphisms.

A *linear extension* $\prec$ of an acyclic category is a total order on its set of objects, such that

$$\mathrm{Mor}(x, y) \neq \emptyset \Longrightarrow x \prec y.$$

Remark 21. We take the term *acyclic category* from [17]. The same name, in other contexts, is given to categories with acyclic nerve. The reader be warned: acyclic categories as defined here must by no means have acyclic nerve.

The data about the cellular collapses that we will perform are stored in so-called *acyclic matchings*.

Definition 50. A *matching* of an acyclic category $\mathcal{C}$ is a set $\mathfrak{M}$ of indecomposable morphisms such that, for every m, $m' \in \mathfrak{M}$, the sources and the targets of m and m' are four distinct objects of $\mathcal{C}$. A *cycle* of a matching $\mathfrak{M}$ is an ordered sequence of morphisms

$$a_1 b_1 a_2 b_2 \cdots a_n b_n$$

where

(1) For all i, $a_i \notin \mathfrak{M}$ and $b_i \in \mathfrak{M}$,

(2) For all i, the targets of a_i and b_i coincide and the sources of a_{i+1} and b_i coincide - as do the sources of a_1 and b_n.

A matching $\mathfrak{M}$ is called *acyclic* if it has no cycles. A *critical element* of $\mathfrak{M}$ is any object of $\mathcal{C}$ that is neither source or target of any $m \in \mathfrak{M}$.

Lemma 51. A matching $\mathfrak{M}$ of an acyclic category $\mathcal{C}$ is acyclic if and only if there is a linear extension of $\mathcal{C}$ where source and target of every $m \in \mathfrak{M}$ are consecutive.

Proof. The proof of Theorem 11.1 of [17] works with mostly only terminological changes. $\square$

A very handy tool for dealing with (and finding) acyclic matchings is the following result, the proof of which follows as an easy exercise by inspection of the definitions and comparison with [17, Theorem 11.10].

Lemma 52 (Patchwork Lemma)**.** Consider a functor of acyclic categories

$$\varphi : \mathcal{C} \to \mathcal{C}'$$

and suppose that for each object c of C' an acyclic matching $\mathfrak{M}_c$ of $\varphi^{-1}(c)$ is given.

Then the matching $\mathfrak{M} := \bigcup_{c \in \mathrm{Ob}\, \mathcal{C}'} \mathfrak{M}_c$ of $\mathcal{C}$ is acyclic.

Proof. Acyclicity of $\mathfrak{M}$ is proved via the linear extension of $\mathcal{C}$ obtained by concatenation of the linear extensions given by the $\mathfrak{M}_c$ on the categories $\varphi(c)$. $\square$

The topological gist of Discrete Morse Theory is the so-called "Fundamental Theorem" (see e.g. [17, 11.2.2]). Here we state the part of it that will be needed below.

Theorem 53. Let $\mathcal{F}$ be the face category of a CW complex X, and let $\mathfrak{M}$ be an acyclic matching of $\mathfrak{M}$. Then X is homotopy equivalent to a CW-complex X' with, for all k, one cell of dimension k for every critical element of $\mathfrak{M}$ of rank k.

Proof. The proof follows the lines of the proof of Theorem 11.13.(b) of [17]. $\square$

Remark 22. If an acyclic matching $\mathfrak{M}$ for a complex X is given, then the boundary maps of the complex X' in Theorem 53 can be explicitely computed via the formula given in [17, §11.2.2].

4 Stratification of the toric Salvetti complex

We now work our way toward proving the minimality of complements of toric arrangements. We start by defining a stratification of the toric Salvetti Complex, in which each stratum corresponds to a local non broken circuit. Then, in the next Section, we will exploit the structure of this stratification to define a perfect acyclic matching on the Salvetti Category.

4.1 Local geometry of complexified toric arrangements

Consider a rank d complexified toric arragement $\mathscr{A} = \{(\chi_1, a_1), \ldots, (\chi_n, a_n)\}$ with $\chi_i(x) = x^{\alpha_i}$ for $\alpha_i \in \mathbb{Z}^d$. As usual we write $K_i = \{x \in T_\Lambda \mid \chi_i(x) = a_i\}$.

We introduce some central hyperplane arrangements we will work with. Consider the arrangement

$$\mathscr{A}_0 = \{H_i = \ker \langle \alpha_i, \cdot \rangle \mid i = 1, \ldots, n\}$$

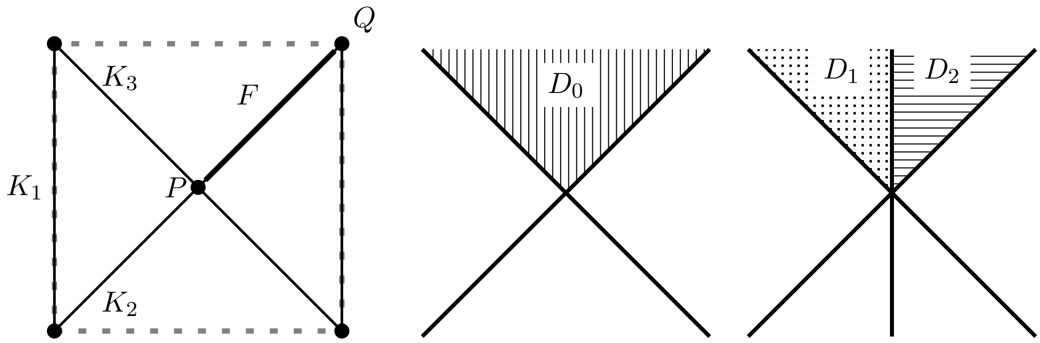
in $\mathbb{R}^d$ and, from now on, fix a chamber $B \in \mathcal{T}(\mathscr{A}_0)$ and a linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$.

Definition 54. For every face $F \in \mathcal{F}(\mathscr{A})$ define the arrangement

$$\mathscr{A}[F] = \{H_i \in \mathscr{A}_0 \mid \chi_i(F) = a_i\}.$$

If $Y \in \mathcal{C}(\mathscr{A})$ define

$$\mathscr{A}[Y] = \{H_i \in \mathscr{A}_0 \mid Y \subseteq K_i\}.$$



(a) A toric arrangement (b) The arrangement $\mathscr{A}[P]$ (c) The arrangement $\mathscr{A}[Q]$

Figure 2: A toric arrangement and its associated hyperplane arrangements

Remark 23. The linear extension $\prec_0$ of $\mathcal{T}(\mathscr{A}_0)_B$ induces as in Proposition 11 linear extensions $\prec_F$ of $\mathcal{T}(\mathscr{A}[F])_{B_F}$ and $\prec_Y$ of $\mathcal{T}(\mathscr{A}[Y])_{B_Y}$, for every $F \in \mathcal{F}(\mathscr{A})$ and every $Y \in \mathcal{C}(\mathscr{A})$.

Moreover, for $F \in \mathcal{F}(\mathscr{A})$ and $C, C' \in \mathcal{T}(\mathscr{A}[F])$ we denote by $S_F(C, C')$ the set of separating hyperplanes of the arrangement $\mathscr{A}[F]$, as introduced in Definition 8.

Definition 55. Given $X \in \mathcal{C}(\mathscr{A})$ let $\widetilde{X} \in \mathcal{L}(\mathscr{A}_0)$ be defined as

$$\widetilde{X} := \bigcap_{X \subseteq K_i} H_i.$$

Definition 56. Let $Y \in \mathcal{C}(\mathscr{A})$ be a layer of $\mathscr{A}$. For $C \in \mathcal{T}(\mathscr{A}[Y])$ let $X(Y, C) \supseteq Y$ be the layer determined by the intersection defined by Lemma 22 from $\prec_Y$. Analogously, for $C \in \mathcal{T}(\mathscr{A}[F])$ let $X(F, C)$ be defined with respect to $\prec_F$.

We write $\widetilde{X}(Y, C)$ and $\widetilde{X}(F, C)$ for the corresponding elements of $\mathcal{L}(\mathscr{A}[Y])$ and $\mathcal{L}(\mathscr{A}[F])$.

Definition 57. Let

$$\mathscr{Y} := \{(Y, C) \mid Y \in \mathcal{C}(\mathscr{A}),\, C \in \mathcal{T}(\mathscr{A}[Y]),\, X(Y, C) = Y\}.$$

For $i = 0, \ldots, d$ let $\mathscr{Y}_i := \{(Y, C) \in \mathscr{Y} \mid \dim(Y) = i\}$.

Example 58. Consider the toric arrangement $\mathscr{A} = \{(x, 1), (xy^{-1}, 1), (xy, 1)\}$ of Figure 2(a). In this and in the following pictures we consider the compact torus $(S^1)^2$ as a quotient of the square. Therefore we draw toric arrangements in a square (pictured with a dashed line), where the opposite sides are identified.

The layer poset consists of the following elements

$$\mathcal{C}(\mathscr{A}) = \{P, Q, K_1, K_2, K_3, (\mathbb{C}^*)^2\}.$$

Figures 2(b) and 2(c) show respectively the arrangements $\mathscr{A}[P]$ and $\mathscr{A}[Q] = \mathscr{A}_0$.

Let $\mathscr{Y}$ as in Definition 57. There is one element $(P, D_0) \in \mathscr{Y}$ and two elements $(Q, D_1), (Q, D_2) \in \mathscr{Y}$. Furthermore we have an element for each 1-dimensional Layer $(K_i, D_i) \in \mathscr{Y}$.

Lemma 59. Let $\mathscr{A}$ be a rank d toric arrangement. For all $i = 0, \dots d$, we have $|\mathscr{Y}_i| = |\mathscr{N}_i|$.

Proof. This follows because for every $i = 0, \dots, d$,

$$|\mathscr{N}_i| = \sum_{\substack{Y \in \mathcal{C}(\mathscr{A}) \\ \dim Y = i}} |\operatorname{nbc}_i(\mathscr{A}[Y])|$$

Every summand on the right hand side counts the number of generators in top degree cohomology or - equivalently - the number of top dimesional cells of a minimal CW-model of the complement of the complexification of $\mathscr{A}[Y]$. By [12, Lemma 4.18 and Proposition 2] these top dimensional cells correspond bijectively to chambers $C \in \mathcal{T}(\mathscr{A}[Y])$ with $X(Y, C) = Y$. Therefore

$$|\mathscr{N}_i| = \sum_{\substack{Y \in \mathcal{C}(\mathscr{A}) \\ \dim Y = i}} |\{C \in \mathcal{T}(\mathscr{A}[Y]) \mid X(Y, C) = Y\}| = |\mathscr{Y}_i|.$$

□

Definition 60. Recall Definition 9. The assignment $(Y, C) \mapsto \mu[\mathscr{A}[Y], \mathscr{A}_0](C)$ defines a function $\xi_0 : \mathscr{Y} \to \mathcal{T}(\mathscr{A}_0)_B$. Choose, and fix, a total order $\dashv$ on $\mathscr{Y}$ that makes this function order preserving.

Remark 24. For $y_1, y_2 \in \mathscr{Y}$, by definition $\xi_0(y_1) \prec_0 \xi_0(y_2)$ implies $y_1 \dashv y_2$.

We now examine the local properties of the ordering $\dashv$.

Definition 61. For $F \in \mathcal{F}(\mathscr{A})$ let $\mathscr{Y}_F := \{(Y, C) \in \mathscr{Y} \mid F \subseteq Y\}$.

Since $F \subseteq Y$ implies $\mathscr{A}[Y] \subseteq \mathscr{A}[F]$, we can define a function $\xi_F : \mathscr{Y}_F \to \mathcal{T}(\mathscr{A}[F])$ by $(Y, C) \mapsto \mu[\mathscr{A}[Y], \mathscr{A}[F]](C)$.

Remark 25. By Lemma 10, $\mu[\mathscr{A}[F], \mathscr{A}_0] \circ \xi_F = \xi_0$ on $\mathscr{Y}_F$. Therefore, for $y_1, y_2 \in \mathscr{Y}_F$, $\xi_F(y_1) \prec_F \xi_F(y_2)$ implies $\xi_0(y_1) \prec_0 \xi_0(y_2)$, and thus $y_1 \dashv y_2$.

Proposition 62. For all $F \in \mathcal{F}(\mathscr{A})$ and every $y = (Y, C) \in \mathscr{Y}_F$,

$$X(F, \xi_F(y)) = Y.$$

Proof. We will use the lattice isomorphisms $\mathcal{L}(\mathscr{A}[F])_{\leq \widetilde{Y}} \simeq \mathcal{L}(\mathscr{A}[Y]) \simeq \mathcal{C}(\mathscr{A})_{\leq Y}$. By definition we have that

$$\xi_F(y) = \mu[\mathscr{A}[Y], \mathscr{A}[F]](C) = \min_{\prec_F}\{K \in \mathcal{T}(\mathscr{A}[F]) \mid K \subseteq C\}$$

and therefore $\mathscr{A}[F]_{\widetilde{Y}} \cap S_F(\xi_F(y), C_1) \neq \emptyset$ for all $C_1 \prec_F \xi_F(y)$, which shows that $\widetilde{Y} \geq \widetilde{X}(F, \xi_F(y))$ in $\mathcal{L}(\mathscr{A}[F])$ and thus $Y \geq X(F, \xi_F(y))$ in $\mathcal{C}(\mathscr{A})$. Now, for every layer Z with $Z < Y$ we have that $\mathscr{A}[Z] \subseteq \mathscr{A}[Y]$. Because by definition $Y = X(Y, C)$, we have $\widetilde{Z} < \widetilde{Y} = \widetilde{X}(Y, C)$ in $\mathcal{L}(\mathscr{A}[Y])$ and so there is $C_2 \prec_Y C$ with $S_Y(C_2, C) \cap A[Y]_{\widetilde{Z}} = \emptyset$.

Let $C_3 := \mu[\mathscr{A}[Y], \mathscr{A}[F]](C_2)$. We have $C_3 \subseteq C_2$ and $\xi_F(y) \subseteq C$, therefore $S_F(C_3, \xi_F(y)) \cap \operatorname{supp}(\widetilde{Z}) = \emptyset$, and $C_3 \prec_F \xi_F(y)$ by $C_2 \prec_Y C$. This means $Z \not\geq X(F, \xi_F(y))$, and the claim follows. ◻

Lemma 63. For $F \in \mathcal{F}(\mathscr{A})$ and $C \in \mathcal{T}(\mathscr{A}[F])$ we have

$$\xi_F(X_C, \sigma_{\mathscr{A}[X_C]}(C)) = C$$

In particular $\xi_F : \mathscr{Y}_F \to \mathcal{T}(\mathscr{A}[F])$ is a bijection.

Proof. Using the definition of ξ_F and Corollary 23 we have

$$\begin{aligned}\xi_F(X_C, \sigma_{\mathscr{A}[X_C]}(C)) &= \mu[\mathscr{A}[X_C], \mathscr{A}[F]](\sigma_{\mathscr{A}[X_C]}(C)) \\ &= \min\{K \in \mathcal{T}(\mathscr{A}[F]) \mid K_{X_C} = C_{X_C}\} = C.\end{aligned}$$

Letting $\beta_F : \mathcal{T}(\mathscr{A}[F]) \to \mathscr{Y}_F$ be defined by $C \mapsto (X_C, \sigma_{\mathscr{A}[X_C]}(C))$, the above means $\xi_F \circ \beta_F = id$, therefore the map ξ_F is surjective. Injectivity of ξ_F amounts now to proving $\beta_F \circ \xi_F = id$, which is an easy check of the definitions. ◻

Corollary 64. For $y_1, y_2 \in \mathscr{Y}_F$, $y_1 \dashv y_2$ if and only if $\xi_F(y_1) \preceq_F \xi_F(y_2)$.

We now relate our constructions to the covering $\mathscr{A}^{\upharpoonright}$ of $\mathscr{A}$ of §2.2.2.

Definition 65. Consider a toric arrangement $\mathscr{A}$ on $T_\Lambda \cong (\mathbb{C}^*)^k$ and a morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A})$. We associate to m a face $F_m \in \mathcal{F}(\mathscr{A}[F])$ as follows.

(a) Fix an $F^{\upharpoonright} \in \mathcal{F}(\mathscr{A}^{\upharpoonright})$ such that $q(F^{\upharpoonright}) = F$.

(b) From Proposition 44 and from the freenees of the action of Λ it follows that there is a unique $G^{\upharpoonright} \in \mathcal{F}(\mathscr{A}^{\upharpoonright})$ such that

$$q(F^{\upharpoonright} \leq G^{\upharpoonright}) = m.$$

(c) Consider the arrangement

$$\mathscr{A}^{\upharpoonright}_{F^{\upharpoonright}} = \{H \in \mathscr{A}^{\upharpoonright} : \quad F^{\upharpoonright} \in H\}.$$

Clearly, up to translation, $\mathscr{A}^{\upharpoonright}_{F^{\upharpoonright}} = \mathscr{A}[F]$ and we can identify the two arrangements.

(d) Define F_m as the face of $\mathscr{A}[F]$ which contains $G^{\upharpoonright}$. That is, in terms of sign vectors and identifying each $H \in \mathscr{A}[F]$ with its unique translate which contains $G^{\upharpoonright}$:

$$\gamma_{F_m} = \gamma_{G^{\upharpoonright}|\mathscr{A}[F]}.$$

In particular, when G is a chamber, then F_m also is.

Remark 26. In order to keep the notation transparent we will often identify a face $F \in \mathcal{F}(\mathscr{A})$, with the corresponing minimal face $F_{id} \in \mathcal{F}(\mathscr{A}[F])$.

Remark 27. Consider a face $F \in \mathcal{F}(\mathscr{A})$ and an element $G^{\upharpoonright} \in \mathcal{F}(\mathscr{A}[F])$. Then there is a unique face $G \in \mathcal{F}(\mathscr{A})$ and a unique morphism $m : F \to G$ such that $G^{\upharpoonright} = i_m(G_{id})$.

Lemma 66. If $m_1 : F_1 \to C_1$ and $m_2 : F_2 \to C_2$ are elements of $\mathrm{Sal}\,\mathscr{A}$ and if there is $l : F_2 \to F_1$, then

$$\pi_{F_1}(m_1) = \pi_{F_1}(m_2) \text{ if and only if } S_{F_2}(F_{l\circ m_1}, F_{m_2}) \cap \mathscr{A}[F_1] = \emptyset$$

Proof. This is a rephrasing of the definitions. □

4.2 Definition of the strata

Definition 67. Define the map $\theta : \mathrm{Sal}(\mathscr{A}) \to \mathscr{Y}$ as follows

$$\theta : (m : F \to C) \mapsto (X(F, F_m), \sigma_{\mathscr{A}[X(F,F_m)]}(F_m))$$

Remark 28. For every object $m : F \to C$ of $\mathrm{Sal}(\mathscr{A})$ we have $\xi_F(\theta(m)) = F_m$.

Lemma 68. For $m : G \to C, m' : G \to C' \in \zeta$, if $\theta(m) \dashv \theta(m')$ then $F_m \prec_G F_{m'}$.

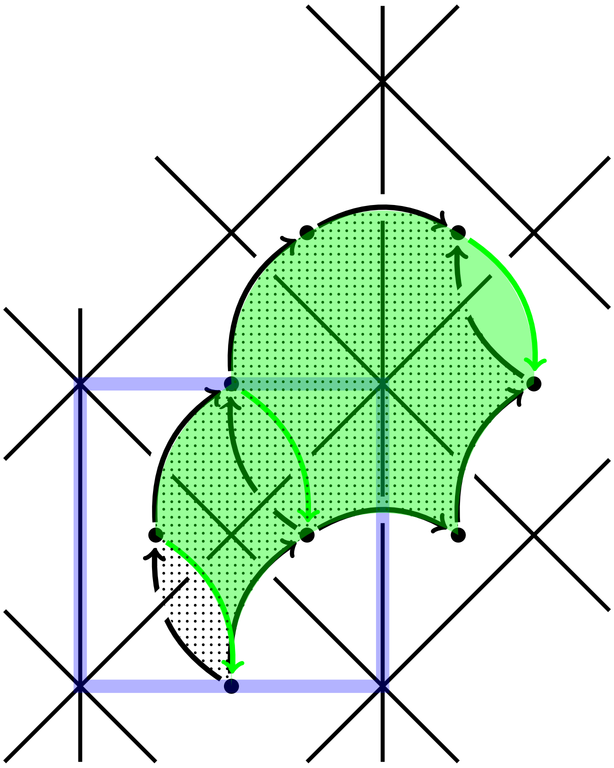
Proof. If $\theta(m) \dashv \theta(m')$, then with Remark 28 and Corollary 64, $F_m = \xi_G(\theta(m)) \prec_G \xi_G(\theta(m')) = F_{m'}$. □

Definition 69. Given a complexified toric arrangement $\mathscr{A}$ on $(\mathbb{C}^*)^d$, we consider the following stratification of $\mathrm{Sal}(\mathscr{A})$ indexed by $\mathscr{Y}$: $\mathrm{Sal}(\mathscr{A}) = \cup_{(Y,C)\in\mathscr{Y}} \mathcal{S}_{(Y,C)}$ where

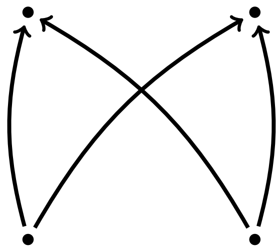
$$\mathcal{S}_{(Y,C)} = \{m \in \mathrm{Sal}(\mathscr{A}) \mid \exists (m \to n) \in \mathrm{Mor}(\mathrm{Sal}(\mathscr{A})),\, n \in \theta^{-1}(Y, C)\}.$$

Moreover, recall the total ordering $\vdash$ on $\mathscr{Y}$ and define

$$\mathcal{N}_y = \mathcal{S}_y \setminus \bigcup_{y' \dashv y} \mathcal{S}_{y'}.$$



(a) Stratification of the toric salvetti complex



(b) $\mathcal{N}_{(K_2,D_2)} = \mathcal{F}(\mathscr{A}^{K_2})^{op} = \mathcal{F}(\mathscr{A}^{K_2})$

Figure 3: Stratification of the toric Salvetti Complex

Example 70. Consider the toric arrangement $\mathscr{A}$ of Figure 2. Figure 3 (a) shows two strata of the stratification on $\mathrm{Sal}\,\mathscr{A}$ of Definition 69.

The stratum $\mathcal{S}_{((\mathbb{C}^*)^2,D)}$ is pictured in dotted black, while the startum $\mathcal{N}_{(K_2,D_2)}$ is pictured in solid green. Thus $\mathcal{N}_{(K_2,D_2)}$ consists of two 1-dimensional layers and two 2-dimensional layers. The category $\mathcal{N}_{(K_2,D_2)}$ is showed in Figure 3 (a) and it is isomorphic to $\mathcal{F}(\mathscr{A}^{K_2})$ (which is self-dual).

5 The topology of the Strata

We now want to show that, for $y \in \mathscr{Y}$, the category $\mathcal{N}_y$ is isomorphic to the face category of a complexified toric arrangement. The main result of this section is the following.

Theorem 71. Consider a complexified toric arrangement $\mathscr{A}$ and for $y = (Y, C) \in \mathscr{Y}$ let $\mathcal{N}_y$ be as in Definition 69. Then there is an isomorphism of acyclic categories

$$\mathcal{N}_{(Y,C)} \cong \mathcal{F}(\mathscr{A}^Y)^{op}$$

The main idea for proving this theorem is to use the 'local' combinatorics of the (hyperplane) arrangements $\mathscr{A}[F]$ to understand the 'global' structure of the strata in $\mathrm{Sal}(\mathscr{A})$. We carry out this 'local-to-global' approach by using the language of diagrams.

Definition 72. Let $\mathscr{A}$ be a complexified toric arrangement. Consider the following diagram of acyclic categories.

$$\mathscr{F} : \mathcal{F}(\mathscr{A})^{op} \to \boldsymbol{AC}; \quad F \mapsto \mathcal{F}(\mathscr{A}[F]);$$
$$(m : F \to G) \mapsto (i_m : \mathcal{F}(\mathscr{A}[G]) \to \mathcal{F}(\mathscr{A}[F]))$$

where for $G' \in \mathcal{F}(\mathscr{A}[G])$ the face $i_m(G') \in \mathcal{F}(\mathscr{A}[F])$ is defined by the following sign vector

$$\gamma_{i_m(G')}(H) = \begin{cases} \gamma_{F_m}(H) & \text{if } H \notin \mathscr{A}[G] \\ \gamma_{G'}(H) & \text{if } H \in \mathscr{A}[G] \end{cases}$$

Example 73. Consider the arrangement $\mathscr{A}$ of Figure 2. Figure 4 illustrates the maps i_m and i_n for the morphisms $m : P \to F$ and $n : Q \to F$.

Lemma 74. Consider the composable morphisms $F \xrightarrow{m} G \xrightarrow{n} K$. Then, with the notation of Definition 65,

$$i_m(G_n) = F_{n \circ m}.$$

Proof. Choose a lift $F^\upharpoonright \in \mathcal{F}(\mathscr{A}^\upharpoonright)$ such that $q(F^\upharpoonright) = F$ and let $G^\upharpoonright \in \mathcal{F}(\mathscr{A}^\upharpoonright)$ the unique face of $\mathscr{A}^\upharpoonright$ such that $q(F^\upharpoonright \leq G^\upharpoonright) = m$. Then $q(G^\upharpoonright) = G$ and there exists a

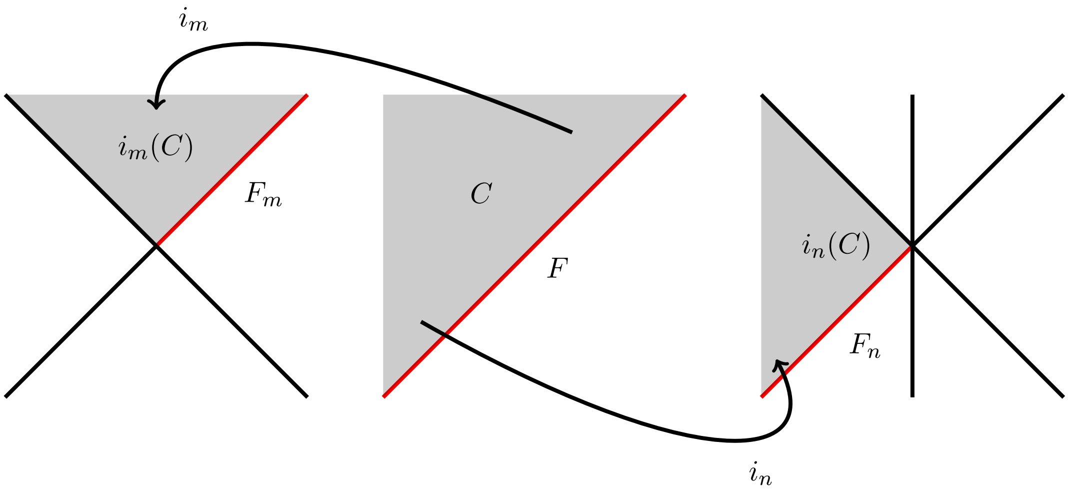


Figure 4: F_m and the map i_m

unique $K^{\upharpoonright} \in \mathcal{F}(\mathscr{A}^{\upharpoonright})$ such that $q(G^{\upharpoonright} \leq K^{\upharpoonright}) = n$. Furthermore $q(F^{\upharpoonright} \leq K^{\upharpoonright}) = n \circ m$. According to Definition 65 we have:

$$\gamma_{i_m(G_n)}(H) = \begin{cases} \gamma_{F_m}(H) & \text{if } H \notin \mathscr{A}[G] \\ \gamma_{G_n}(H) & \text{if } H \in \mathscr{A}[G] \end{cases} = \begin{cases} \gamma_{G^{\upharpoonright}}(H) & \text{if } H \notin \mathscr{A}[G] \\ \gamma_{K^{\upharpoonright}}(H) & \text{if } H \in \mathscr{A}[G] \end{cases} \tag{6}$$

In terms of sign vectors, the property $G^{\upharpoonright} \leq K^{\upharpoonright}$ translates to the following.

$$\text{For all } H \in \mathscr{A}^{\upharpoonright}: \quad \gamma_{G^{\upharpoonright}}(H) \neq 0 \Longrightarrow \gamma_{G^{\upharpoonright}}(H) = \gamma_{K^{\upharpoonright}}(H).$$

In particular $H \notin \mathscr{A}[G]$ implies $\gamma_{G^{\upharpoonright}}(H) = \gamma_{K^{\upharpoonright}}(H)$ and therefore from Equation (6) we get

$$\gamma_{i_m(G_n)}(H) = \gamma_{K^{\upharpoonright}}(H) \quad \forall H \in \mathscr{A}[F], \qquad \text{which means} \qquad i_m(G_n) = F_{m \circ n}.$$

$\square$

Lemma 75.

$$\operatorname{colim} \mathscr{F} = \mathcal{F}(\mathscr{A})$$

Proof. Let us first recall the usual construction of colimits. The category $\operatorname{colim} \mathscr{F}$ consists of the object set

$$\operatorname{Obj}(\operatorname{colim} \mathscr{F}) = \bigsqcup_{F \in \mathcal{F}(\mathscr{A})^{op}} \operatorname{Obj}(\mathcal{F}(\mathscr{A}[F]))/\sim.$$

Where the equivalence relation $\sim$ is generated by the relations $G' \sim \mathscr{F}(m)(G')$ for all morphisms $(m : F \to G) \in \mathrm{Mor}(\mathcal{F}(\mathscr{A}))$ and all $G' \in \mathrm{Obj}(\mathcal{F}(\mathscr{A}[G]))$. The morphisms of $\operatorname{colim} \mathscr{F}$ are similarly given by

$$\mathrm{Mor}(\operatorname{colim} \mathscr{F}) = \bigsqcup_{F \in \mathcal{F}(\mathscr{A})^{op}} \mathrm{Mor}(\mathcal{F}(\mathscr{A}[F]))/\sim$$

with the relation $\sim$ generated by $n \sim \mathscr{F}(m)(n)$ for every morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A})$ and every morphism $n : G' \to G''$ of $\mathcal{F}(\mathscr{A}[G])$.

Equivalence classes with respect to these equivalence relations will be denoted by $[\![\,\cdot\,]\!]$, to avoid confusion with the square brackets used to identify elements of the Salvetti complex.

Next we construct an isomorphism $\Phi : \mathcal{F}(\mathscr{A}) \to \operatorname{colim} \mathscr{F}$. Consider an object $F \in \mathcal{F}(\mathscr{A})$ and define $\Phi(F) = [\![F_{id}]\!]$, where F_{id} is a face in $\mathcal{F}(\mathscr{A}[F])$. Consider now a morphism $m : F \to G$ in $\mathcal{F}(\mathscr{A})$ and define

$$\Phi(m) = [\![F_{id} \leq F_m]\!].$$

The bijectivity of Φ is easily seen. We only need to show the functoriality of Φ. Consider the composable morphisms $F \overset{m}{\to} G \overset{n}{\to} H$. Using Lemma 74 we get

$$\begin{aligned}
\Phi(n) \circ \Phi(m) &= [\![G_{id} \leq G_n]\!] \circ [\![F_{id} \leq F_m]\!] \\
&= [\![\mathscr{F}(m)(G_{id} \leq G_n)]\!] \circ [\![F_{id} \leq F_m]\!] = [\![i_m(G_{id}) \leq i_m(G_n)]\!] \circ [\![F_{id} \leq F_m]\!] \\
&= [\![F_m \leq F_{n \circ m}]\!] \circ [\![F \leq F_m]\!] = [\![F \leq F_{n \circ m}]\!] = \Phi(n \circ m).
\end{aligned}$$

□

Definition 76.

$$\begin{aligned}
\mathscr{D} = \mathscr{D}(\mathscr{A}) : \mathcal{F}(\mathscr{A})^{op} &\to \boldsymbol{AC}; \\
F &\mapsto \mathrm{Sal}(\mathcal{A}[F]); \\
(m : F \to G) &\mapsto j_m : \mathrm{Sal}(\mathscr{A}[G]) \hookrightarrow \mathrm{Sal}(\mathscr{A}[F])
\end{aligned}$$

where $j_m([G, C]) = [i_m(G), i_m(C)]$.

Lemma 77.

$$\operatorname{colim} \mathscr{D}(\mathscr{A}) = \mathrm{Sal}(\mathscr{A})$$

Remark 29. Using Remark 27 we have that every element $\varepsilon \in \operatorname{colim} \mathscr{D}(\mathscr{A})$ has a (unique) representant $[F, C] \in \mathcal{S}(\mathscr{A}[F])$ such that for every other representant $[G, K]$ with $\varepsilon = [\![G, K]\!]$ there is a unique morphism $m : F \to G^{\downarrow}$ with $[G, K] = [F_m, i_m(C)]$.

Proof of Lemma 77. The proof follows the outline of the proof of Lemma 75, the isomorphism $\Psi : \mathrm{Sal}(\mathscr{A}) \to \operatorname{colim} \mathscr{D}$ being defined as follows. For an object $m : F \to C$ of $\mathrm{Sal}(\mathscr{A})$ (i.e. a morphism of $\mathcal{F}(\mathscr{A})$) define $\Psi(m) = \Phi(m) = [\![F_{id}, F_m]\!]$. For a morphism (n, m_1, m_2) of $\mathrm{Sal}(\mathscr{A})$ with $m_i : F_i \to C_i$ and $n : F_2 \to F_1$ define

$$\Psi(n, m_1, m_2) = [\![\mathscr{D}(n)([(F_1)_{id}, F_{m_1}]) \leq [(F_2)_{id}, F_{m_2}]]\!] =$$
$$[\![[i_n((F_1)_{id}), i_n(F_{m_1})] \leq [(F_2)_{id}, F_{m_2}]]\!] = [\![[F_n, F_{m_1 \circ n}] \leq [(F_2)_{id}, F_{m_2}]]\!],$$

where in the last equality we used Lemma 74. □

Remark 30. Note that, given any chamber C of $\mathscr{A}[G]$, $\mathcal{S}(\mathscr{A}[G])_C \hookrightarrow \mathcal{S}(\mathscr{A}[F])_{C'} \subseteq \mathcal{S}(\mathscr{A}[F])$ if and only if $S(i_m(C), C') \cap \mathscr{A}[G] = \emptyset$.

Lemma 78. Let $m : F \to G$ be a morphism of $\mathcal{F}(\mathscr{A})$ and consider an $(Y, C) \in \mathscr{Y}_F$. Then the inclusion $j_m : \mathrm{Sal}(\mathscr{A}[G]) \to \mathrm{Sal}(\mathscr{A}[F])$ restricts to an inclusion

$$j_m : \mathcal{S}_{\xi_G(Y,C)} \to \mathcal{S}_{\xi_F(Y,C)}.$$

Proof. We only need to show that $S(i_m(\xi_G(Y,C)), \xi_F(Y,C)) \cap \mathscr{A}[G] = \emptyset$. Let $H \in \mathscr{A}[G]$, then

$$\gamma_{i_m(\xi_G(Y,C))}(H) = \gamma_{\xi_G(Y,C)}(H) = \gamma_{\xi_F(Y,C)}(H) \Longrightarrow H \notin S(i_m(\xi_G(Y,C)), \xi_F(Y,C))$$

where the last equality follows from the fact that $\xi_F(Y,C) \subseteq \xi_G(Y,C)$. □

Lemma 78 allows us to state the following definition.

Definition 79. For any $(Y, C) \in \mathscr{Y}$ let

$$\mathscr{E}_{(Y,C)} : \mathcal{F}(\mathscr{A}^Y)^{op} \to \boldsymbol{AC}; \quad F \mapsto \mathcal{S}(\mathscr{A}[F])_{\xi_F(Y,C)}; \quad (m : F \to G) \mapsto (j_m)_{|\mathscr{E}_{(Y,C)}(G)}$$

Lemma 80. Let $(Y, C) \in \mathscr{Y}$, then

$$\operatorname{colim} \mathscr{E}_{(Y,C)} = \mathcal{S}_{(Y,C)}$$

Proof. We consider the isomorphism $\Psi : \mathrm{Sal}(\mathscr{A}) \to \operatorname{colim} \mathscr{D}$ of Lemma 77. We want to show that $\Psi(\mathcal{S}_{(Y,C)}) = \operatorname{colim} \mathscr{E}_{(Y,C)}$.

Let $[\![G, K]\!] \in \operatorname{colim} \mathscr{E}_{(Y,C)}$, then (recall Remark 29) there is a morphism of $\mathcal{F}(\mathscr{A})$ $m : F \to G$ such that $[F_m, i_m(K)] \in \mathcal{S}_{\xi_F(Y,C)} \subseteq \mathrm{Sal}(\mathscr{A}[F])$, i.e.

$$[F_m, i_m(K)] \leq [F, \xi_F(Y,C)].$$

Taking the preimage through Ψ of this relation we get a morphism

$$\Psi^{-1}([\![G, K]\!]) \to \Psi^{-1}([\![F, \xi_F(Y,C)]\!]) \in \mathrm{Mor}(\mathrm{Sal}(\mathscr{A})).$$

Now, using Proposition 62 we have

$$\theta(\Psi^{-1}([\![F, \xi_F(Y,C)]\!])) = (X(F, \xi_F(Y,C)), \sigma_{\mathscr{A}[Y]}\xi_F(Y,C))$$
$$= (Y, \sigma_{\mathscr{A}[Y]}\mu[\mathscr{A}[Y], \mathscr{A}[F]]C) = (Y,C).$$

Therefore $\Psi^{-1}([\![G,K]\!]) \in \mathcal{S}_{(Y,C)}$, so $[\![G,K]\!] \in \Psi\left(\mathcal{S}_{(Y,C)}\right)$ and we have proved that $\operatorname{colim} \mathscr{E}_{(Y,C)} \subseteq \Psi(\mathcal{S}_{(Y,C)})$.

To prove the converse inclusion, let $(m : G \to K) \in \mathcal{S}_{(Y,C)}$. Then there is a morphism $(h,m,n) : m \to n \in \operatorname{Mor}(\operatorname{Sal}(\mathscr{A}))$ with $n : F \to K'$, $h : F \to G$ and $\theta(n) = (Y,C)$. In particular, in view of Remark 28, we get $F_n = \xi_F(\theta(n)) = \xi_F(Y,C)$.

Applying Ψ to the morphism (h,m,n), in $\operatorname{Sal}(\mathscr{A}[F])$ we obtain

$$j_n([G, G_m]) \leq [F, F_n] = [F, \xi_F(Y,C)], \text{ thus } j_n([G, G_m]) \in \mathcal{S}_{\xi_F(Y,C)},$$

and we conclude that

$$\Psi(m) = [\![G, G_m]\!] = [\![j_n([G, G_m])]\!] \in \operatorname{colim} \mathscr{E}_{(Y,C)},$$

proving $\Psi(\mathcal{S}_{(Y,C)}) \subseteq \operatorname{colim} \mathscr{E}_{(Y,C)}$. □

Definition 81.

$$\mathscr{G}_{(Y,C)} : \mathcal{F}(\mathscr{A}^Y)^{op} \to \boldsymbol{AC}; \quad F \mapsto \mathcal{N}_{\xi_F(Y,C)}; \quad (m : F \to G) \mapsto (j_m)_{|\mathscr{G}_{(Y,C)}(G)}$$

Remark 31. To prove that the diagram $\mathscr{G}_{(Y,C)}$ is well defined, we have to show that for every morphism $m : F \to G$ of $\mathcal{F}(\mathscr{A}^Y)$

$$j_m(\mathcal{N}_{\xi_G(Y,C)}) \subseteq \mathcal{N}_{\xi_F(Y,C)}. \tag{7}$$

This follows because by Proposition 62 we have $X(F, \xi_F(Y,C)) = Y$, and thus with [12, Lemma 4.18] we can rewrite

$$\mathcal{N}_{\xi_F(Y,C)} = \{[G,K] \in \operatorname{Sal}(\mathscr{A}[F]) \mid G \in \mathcal{F}(\mathscr{A}[F]^{\widetilde{Y}}),\ K_G = \xi_F(Y,C)_G\}.$$

Now let $[G', C'] \in \mathcal{N}_{\xi_G(Y,C)}$. Then since $G' \subseteq \widetilde{Y}$ we have $i_m(G') \in \mathcal{F}(\mathscr{A}[F]^{\widetilde{Y}})$, and from $\xi_F(Y,C) \subseteq \xi_G(Y,C)$ we conclude $i_m(C')_{G'} = \xi_F(Y,C)_{G'}$. Therefore $j_m([G', C']) = [i_m(G'), i_m(C')] \in \mathcal{N}_{\xi_F(Y,C)}$, and the inclusion (7) is proved.

Lemma 82.

$$\operatorname{colim} \mathscr{G}_{(Y,C)} = \mathcal{N}_{(Y,C)}$$

Proof. First, we prove that $\operatorname{colim} \mathscr{G}_{(Y,C)} \subseteq \mathcal{N}_{(Y,C)}$. For this, let $[\![F, K]\!] \in \operatorname{colim} \mathscr{G}_{(Y,C)}$ and suppose $[\![F, K]\!] \notin \mathcal{N}_{(Y,C)}$. Then $[\![F, K]\!] \in \operatorname{colim} \mathscr{E}_{(Y',C')}$ for some $(Y', C') < (Y, C)$. Now, since $[\![F, K]\!] \in \operatorname{colim} \mathscr{G}_{(Y,C)}$ there exist a point $P \in \mathcal{F}(\mathscr{A})$ and a morphism $m : P \to F$ with $[P_m, i_m(K)] \in \mathcal{N}_{\xi_P(Y,C)}$. Therefore, in $\mathscr{A}[P]$ we have $[P_m, i_m(K)] \leq [P, \xi_P(Y, C)]$, which implies $K_{P_m} = \xi_P(Y, C)_{P_m}$, and thus $K = \sigma_{\mathscr{A}[F]}(K_{P_m}) = \xi_F(Y, C)$.

Similarly, since $[\![F, K]\!] \in \operatorname{colim} \mathscr{E}_{(Y',C')}$ there is a point $Q \in \mathcal{F}(\mathscr{A})$ and a morphism $n : Q \to F$ with $[Q_n, i_n(K)] \in \mathcal{S}_{\xi_Q(Y',C')}$. Then, as above, $K = \xi_F(Y', C')$

From the bijectivity proven in Lemma 63 we conclude $(Y, C) = (Y', C')$, which contradicts $(Y', C') < (Y, C)$, proving that $[\![F, K]\!] \in \mathcal{N}_{(Y,C)}$, as desired.

The other inclusion is easier. Suppose $[F, K] \in \mathcal{N}_{(Y,C)} \setminus \operatorname{colim} \mathscr{G}_{(Y,C)}$. Then $[F, K] \in \mathcal{S}_{\xi_P(Y',C')}$ for some point $P \in \mathcal{F}(\mathscr{A})$ and some $(Y', C') < (Y, C)$. But then $[F, K] \in \operatorname{colim} \mathscr{E}_{(Y',C')} \Rightarrow [F, K] \notin \mathcal{N}_{(Y,C)}$. □

Lemma 83. There is an equivalence of diagrams

$$\mathscr{G}_{(Y,C)} \cong \mathscr{F}(\mathscr{A}^Y)^{op}$$

Proof. For each $F \in \mathcal{F}(\mathscr{A}^Y)$ define the isomorphisms $\mathscr{G}_{(Y,C)}(F) \to \mathscr{F}(\mathscr{A}^Y)^{op}(F)$ as follows

$$\mathscr{G}_{(Y,C)}(F) = \mathcal{N}_{\xi_F(Y,C)} \cong \mathcal{F}(\mathscr{A}[F]^{\widetilde{Y}})^{op} = \mathcal{F}(\mathscr{A}^Y[F])^{op} = \mathscr{F}(\mathscr{A}^Y)^{op}(F).$$

Where the isomorphism in the middle comes from Theorem 25.

It can be easily checked that these isomorphisms are indeed morphisms of diagrams. □

As a consequence of Lemma 83 we can write the following.

Proof of Theorem 71.

$$\mathcal{N}_{(Y,C)} = \operatorname{colim} \mathscr{G}_{(Y,C)} \cong \operatorname{colim} \mathscr{F}(\mathscr{A}^Y)^{op} = \mathcal{F}(\mathscr{A}^Y)^{op}.$$

□

6 Minimality of toric arrangements

6.1 Perfect matchings for the compact torus

Let $\mathscr{A}$ be a complexified toric arrangement in T_Λ and choose a point $P \in \max \mathcal{C}(\mathscr{A})$. Up to a biholomorphic transformation we may suppose that P is the origin of the torus.

Let then $(\chi_1, a_1), \ldots, (\chi_d, a_d) \in \mathscr{A}$ be such that $\alpha_1, \ldots, \alpha_d$ are ($\mathbb{Q}$-) linearly independent. For $i = 1, \ldots, d$ let H_i^1 denote the hyperplane of $\mathscr{A}^\upharpoonright$ lifting K_i at the

origin of $\hom(\Lambda, \mathbb{R}) \simeq \mathbb{R}^d$. We identify for ease of notation $\Lambda \simeq \mathbb{Z}^d \subseteq \mathbb{R}^d$, and in particular think of α_i as the normal vector to H_i^1.

For $j \in [d]$ we consider the rank $j-1$ lattice

$$\Lambda_j := \mathbb{Z}^d \cap \bigcap_{i \geq j} H_i^1$$

Lemma 84. There is a basis $u_1, \ldots, u_d$ of Λ such that for all $i = 1, \ldots, d$, the elements $u_1, \ldots, u_{i-1}$ are a basis of Λ_i.

Proof. The proof is by repeated application of the Invariant Factor Theorem, e.g. [4, Theorem 16.18], to the free $\mathbb{Z}$-submodule Λ_j of Λ_{j-1}. □

Let $(H_i^1)^+ := \{x \in \mathbb{R}^d \mid \langle x, \alpha_i \rangle \geq 0\}$.

Remark 32. In particular, $u_i \notin H_i^1$, hence $u_i(H_i^1) \neq H_i^1$. Moreover, without loss of generality we may suppose $u_i \in (H_i^1)^+$.

The lattice Λ acts on $\mathbb{R}^d$ by translations. Given $u \in \Lambda$, let the corresponding translation be

$$t_u : \mathbb{R}^d \to \mathbb{R}^d; \qquad x \mapsto t_u(x) := x + u.$$

Corollary 85. For all $x \in \mathbb{R}^d$ and all $i < j \in [d]$, $\langle t_{u_i}(x), \alpha_{d-j} \rangle = \langle x, \alpha_{d-j} \rangle$.

Proof. We have $u_i \in \Lambda_j \subseteq H_{d-j}^1$, therefore $\langle u_i, \alpha_{d-j} \rangle = 0$ and thus

$$\langle t_{u_i}(x), \alpha_{d-j} \rangle = \langle x + u_i, \alpha_{d-j} \rangle = \langle x, \alpha_{d-j} \rangle + \langle u_i, \alpha_{d-j} \rangle = \langle x, \alpha_{d-j} \rangle + 0.$$

□

For $i = 1, \ldots, d$ let $(H_i^2)^+ := t_{u_i}((H_i^1)^+)$, and define

$$Q := \bigcap_{i=1}^{d} [(H_i^1)^+ \setminus (H_i^2)^+].$$

Lemma 86. The region Q is a fundamental region for the action of Λ on $\mathbb{R}^d$.

Proof. For $i = 1, \ldots, d$, write

$$l_i := \langle u_i, \alpha_i \rangle.$$

Then, $Q = \{x \in \mathbb{R}^d \mid 0 \leq \langle x, \alpha_i \rangle < l_i \text{ for all } i = 1, \ldots, d\}$. It is clear that Q can contain at most one point for each orbit of the action of λ.

Now choose and fix an $x \in \mathbb{R}^d$. We want to construct an $y \in Q$ such that $x \in y + \Lambda$.

To this end write $x_0 := x$ and let $\lambda_d := \lfloor \langle x_0, \alpha_d \rangle / l_d \rfloor$. Then let

$$x_1 := x_0 - \lambda_d u_d, \text{ thus } 0 \leq \langle x_1, \alpha_d \rangle < l_d$$

For every $i \in \{1, \dots d-1\}$ let now $\lambda_{d-i} := \lfloor \langle x_i, \alpha_{d-i} \rangle / l_{d-i} \rfloor$.

Then set $x_{i+1} := x_i - \lambda_{d-i} u_{d-i}$, so that

$$0 \leq \langle x_{i+1}, \alpha_{d-i} \rangle < l_{d-i}$$

and so, by Corollary 85, for every $j < i$:

$$\langle x_{i+1}, \alpha_{d-j} \rangle = \langle t_{u_{d-i}}^{-\lambda_{d-i}} \cdots t_{u_{d-j-1}}^{-\lambda_{d-j-1}}(x_{j+1}), \alpha_{d-j} \rangle = \langle x_{j+1}, \alpha_{d-j} \rangle \in [0, l_{d-j}[.$$

After d steps, we will have reached x_d, with

$$0 \leq \langle x_d, \alpha_i \rangle < l_i \text{ for all } i = 1, \dots, d.$$

Hence $y := x_d \in Q$ is the required point because, putting $u := \sum_{i=1}^{d} \lambda_i u_i$, we have by construction $x_d = t_{-u}(x)$ and so $x = t_u(y) \in y + \Lambda$. □

Definition 87. Let $\mathscr{A}$ be a rank d toric arrangement, and let $\mathcal{B}_d$ be the 'boolean poset on d elements', i.e., the acyclic category on the subsets of $[d]$ with the inclusion morphisms. Since $\mathcal{B}_d$ is a poset, the function

$$\mathrm{Ob}(\mathcal{F}(\mathscr{A})) \to \mathrm{Ob}(\mathcal{B}_d), \qquad F \mapsto \{i \in [d] \mid F \subseteq K_i\}$$

induces a well defined functor of acyclic categories

$$\mathcal{I} : \mathcal{F}(\mathscr{A}) \to \mathcal{B}_d^{op}.$$

For every $I \subseteq [d]$ define the category

$$\mathcal{F}_I := \mathcal{I}^{-1}(I)$$

Lemma 88. For all $I \subseteq [d]$, the subcategory $\mathcal{F}_I$ is a poset admitting an acyclic matching with only one critical element (in top rank).

Fix $I \subset [d]$, let $k := |I|$.

We consider

$$Q_I := Q \cap \big(\bigcap_{i \in I} H_i^1 \big) \setminus \bigcup_{j \notin I} \big(H_j^1 \cup H_j^2 \big).$$

The set $\mathscr{B} := \{H \cap X \mid H \in \mathscr{A}^{\upharpoonright}, H \cap Q \neq \emptyset\}$ is a finite arrangement of affine hyperplanes in the affine hull X of Q_I. This arrangement determines a (regular) polyhedral decomposition $\mathcal{D}(\mathscr{B})$ of $\mathbb{R}^{d-k}$ that coincides with $\mathcal{D}(\mathscr{A}^{\upharpoonright}{}_X)$ on Q.

The exponential covering of Section 2.2.2 maps Q_I homeomorphically to its image, hence $\mathcal{F}_I$ is the face category of the set of cells of the decomposition of Q_I by $\mathcal{D}(\mathscr{B})$. Regularity of $\mathcal{D}(\mathscr{B})$ implies that $\mathcal{F}_I$ is a poset. Indeed, if $\mathcal{D}(\mathscr{B})^\vee$ is the (regular) CW-decomposition dual to the one induced by $\mathscr{B}$, then $\mathcal{F}_I^{op}$ is the poset of cells of the subcomplex Y_I that is entirely contained in Q_I.

Let $\mathcal{Q}$ be the subdivision of the closure $\overline{Q_I}$ induced by $\mathscr{B}$.

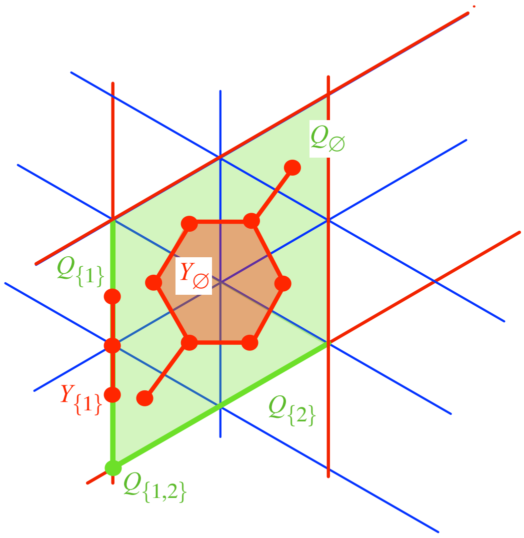


Figure 5: The case of the toric Weyl arrangement of Type A_2

Lemma 89. The complex $\mathcal{Q}$ is shellable.

Proof. Coning the arrangement $\mathscr{B}$ (as in [23, Definition 1.15]) we obtain a central arrangement $\widehat{\mathscr{B}} = \{\widehat{H} \mid H \in \mathscr{B}\}$ which subdivides the unit sphere into a regular cell complex $\mathcal{K}$. Then, $\mathcal{Q}$ is isomorphic to the subcomplex of $\mathcal{K}$ given by

$$\bigcap_{i \notin I} \widehat{H_i^1}^{+} \cap \bigcap_{i \notin I} \widehat{H_i^2}^{-}$$

which, by [2, Proposition 4.2.6 (c)], is shellable. □

Proof of Lemma 88. The pseudomanifold $\mathcal{Q}$ is constructible because it is shellable. With [1, Theorem 4.1], it is also endo-collapsible, i.e., it admits an acyclic matching where the critical cells are precisely the cells on the boundary plus one single cell in the interior of $\mathcal{Q}$. But this restricts to an acyclic matching of the subposet $\mathcal{F}_I \subseteq \mathcal{F}(\mathcal{Q})$ with exacly one critical cell.

In turn this gives an acyclic matching of $\mathcal{F}_I^{op}$ with exactly one critical cell. Since $\mathcal{F}_I^{op}$ is the face poset of the CW-complex Y_I, the critical cell must be in bottom rank - thus in top rank of $\mathcal{F}_I$, as required. □

Proposition 90. For any complexified toric arrangement $\mathscr{A}$, the acyclic category $\mathcal{F}(\mathscr{A})$ admits a perfect acyclic matching.

Proof. Let $\mathscr{A}$ be of rank d. The proof is a straightforward application of the Patchwork Lemma 52 in order to merge the 2^d acyclic matchings described in Lemma 88 along the map $\mathcal{I}$ of Definition 87. The resulting 'global' acyclic matching has 2^d critical elements and is thus perfect. □

6.2 Perfect matchings for the toric Salvetti complex

Let $\mathscr{A}$ be a (complexified) toric arrangement.

Proposition 91. The Salvetti Category $\operatorname{Sal}\mathscr{A}$ admits a perfet acyclic matching.

Proof. Let P denote the acyclic category given by the $|\mathscr{Y}|$-chain. We define a functor of acyclic categories

$$\varphi : \operatorname{Sal}\mathscr{A} \to P; \quad m \mapsto (Y, C) \text{ for } m \in \mathcal{N}_{(Y,C)}$$

and we have an isomorphism of acyclic categories $\varphi^{-1}((Y,C)) = \mathcal{N}_{(Y,C)} \simeq \mathcal{F}(\mathscr{A}_Y)$. Then, by Proposition 90, $\varphi^{-1}((Y,C))$ has an acyclic matching with $2^{d-\operatorname{rk} X}$ critical cells.

An application of the Patchwork Lemma 52 gives then an acyclic matching on $\operatorname{Sal}(\mathscr{A})$ with

$$\sum_j |\mathscr{Y}_j| 2^{d-j} = \sum_j |\mathscr{N}_j| 2^{d-j} = P_{\mathscr{A}}(1)$$

critical cells, where the first equality is given by Lemma 59. This matching is thus perfect. □

Corollary 92. The complement $M(\mathscr{A})$ is a minimal space.

Corollary 93. The homology and cohomology groups $H_k(M(\mathscr{A}), \mathbb{Z})$, $H^k(M(\mathscr{A}), \mathbb{Z})$ are torsion free for all k.

7 Minimality of affine arrangements

After the existence proofs of Dimca and Papadima in [13] and by Randell in [24], the first step towards an explicit characterization of the minimal model was taken by Yoshinaga [27] who, for complexified arrangements, identified the cells of the minimal complex using their incidence with a general position flag in real space and studied their incidence and boundary maps. Salvetti and Settepanella [26] obtained a complete description of the minimal complex by using a 'polar ordering' determined by a general position flag to define a discrete Morse vector field on the Salvetti complex - a combinatorial model of the homotopy type of the complement of a complexified affine arrangement - in order to collapse it to a minimal CW-complex.

The goal of this section is to extend to affine complexified hyperplane arrangements the idea of [12], in order to obtain a minimal complex that is defined only in terms of the arrangement's (affine) oriented matroid.

Consider a finite affine complexified arrangement $\mathscr{A} = \{K_1, \ldots, K_n\}$. Define the central arrangements $\mathscr{A}_0$ and $\mathscr{A}[F]$ for $F \in \mathcal{F}(\mathscr{A})$ in analogy to those of Section 4.1. Choose a base chamber $B \in \mathcal{T}(\mathscr{A}_0)$, fix a total ordering $\prec_0$ on $\mathscr{A}_0$ and define $\prec_F, \prec_Y$ for $F \in \mathcal{F}(\mathscr{A}), Y \in \mathcal{L}(\mathscr{A})$ as in Section 4.1. Moreover, let $\mathscr{Y}$ be as in Definition 57.

Remark 33. Notice that, given the affine oriented matroid of $\mathscr{A}$, the oriented matroid of $\mathscr{A}_0$ can be recovered without referring to the geometry. For instance, the tope poset of $\mathscr{A}_0$ can be defined in terms of the tope poset of $\mathscr{A}$ based at any unbounded chamber.

Lemma 94. Let $\mathscr{A}$ be a finite complexified affine hyperplane arrangement, and $\mathscr{Y}$ as above, then

$$|\mathscr{Y}| = \sum_{k \in \mathbb{N}} \operatorname{rk} H^k(M(\mathscr{A}); \mathbb{Z})$$

Proof. As in Lemma 59, applying [12, Lemma 4.18 and Proposition 2] we have

$$|\{C \in \mathcal{T}(\mathscr{A}[Y]) \mid X(Y, C) = Y\}| = \operatorname{rk} H^{\operatorname{codim} Y}(M(\mathscr{A}_Y); \mathbb{Z}) \quad \forall Y \in \mathcal{L}(\mathscr{A}).$$

Applying Theorem 15 we get the claim. $\square$

We now define the analogue of the map θ of Definition 67.

Definition 95. Let $F, G \in \mathcal{F}(\mathscr{A})$ with $F \subseteq G$ and identify

$$\mathscr{A}[F] = \mathscr{A}_F = \{H \in \mathscr{A} \mid F \subseteq H\},$$

in particular we have an inclusion $\mathscr{A}[G] \subseteq \mathscr{A}[F]$. Define the map $i_{F \leq G} : \mathcal{F}(\mathscr{A}[G]) \to \mathcal{F}(\mathscr{A}[F])$ as follows

$$\gamma_{i_{F \leq G}(J)}(H) = \begin{cases} \gamma_G(H) & \text{if } H \in \mathscr{A}[G] \\ \gamma_F(H) & \text{if } H \notin \mathscr{A}[G], \end{cases} \qquad \forall J \in \mathcal{F}(\mathscr{A}[G]).$$

As above, the map $i_{F \leq G}$ induces a function $j_{F \leq G} : \operatorname{Sal}(\mathscr{A}[F]) \to \operatorname{Sal}(\mathscr{A}[G])$.

Theorem 96 (Lemma 3.2.8 and Theorem 4.2.1 of [11])**.** The assignment $\mathscr{E} : \mathcal{F}(\mathscr{A}) \to AC^{op}$, $\mathscr{E}(F) := \operatorname{Sal}(\mathscr{A}[F])$, $\mathscr{E}(F \leq G) = j_{F \leq G}$ defines a diagram of posets such that $\operatorname{colim} \mathscr{E}$ is poset isomorphic to $\operatorname{Sal}(\mathscr{A})$.

The stratification of $\operatorname{Sal}(\mathscr{A})$ is also defined along the lineas of the preceding sections.

Definition 97. Define the map $\theta : \mathrm{Sal}(\mathscr{A}) \to \mathscr{Y}$ as follows

$$\theta([F,C]) = (X(F, i_{F\le G}(G)), \sigma_{\mathscr{A}[X(F,i_{F\le G}(G))]}(G)).$$

where we identified $G = \min \mathcal{L}(\mathscr{A}[G])$.

Definition 98. Let $\mathscr{A}$ be a finite complexified affine hyperplane arrangement and define a total ordering $\dashv$ on $\mathscr{Y}$ as in Definition 60. Define:

$$\mathcal{S}_{(Y,C)} = \left\{ [F,C] \in \mathrm{Sal}(\mathscr{A}) \,\middle|\, \begin{array}{c} \text{there is } [G,K] \in \mathrm{Sal}(\mathscr{A}) \text{ with} \\ [F,C] \le [G,K] \text{ and } \theta([G,K]) = (Y,C) \end{array} \right\}$$

$$\mathcal{N}_{(Y,C)} = S_{(Y,C)} \setminus \bigcup_{(Y',C') \dashv (Y,C)} \mathcal{S}_{(Y',C')}.$$

The arguments of Section 5 can now be adapted to the affine case, obtaining the following analogon of Theorem 71.

Theorem 99. Let $\mathscr{A}$ be a finite complexified affine hyperplane arrangement. There is an isomorphism of posets

$$\mathcal{N}_{(Y,C)} \cong \mathcal{F}(\mathscr{A}^Y)^{op} \qquad \forall (Y,C) \in \mathscr{Y}.$$

The analogon of Proposition 90 is proved in [2, 4.5.7, 4.5.8], from which it follows that the poset $\mathcal{N}^{op}_{(Y,C)}$ is shellable, and therefore $\mathcal{N}_{(Y,C)}$ admits an acylic matching with one critical cell in top dimension. Applying the Patchwork Lemma as in Proposition 91 we obtain an acyclic matching $\mathfrak{M}$ of $\mathrm{Sal}(\mathscr{A})$ with the minimal number of critical cells. We summarize.

Proposition 100. Let $\mathscr{A}$ be a finite complexified affine hyperplane arrangement. The oriented matroid data of $\mathscr{A}$ define a discrete Morse function on $\mathrm{Sal}(\mathscr{A})$ that collapses the Salvetti complez to a minimal complex. In particular, the complement $M(\mathscr{A})$ is a minimal space.

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